(* COUNTS *)

26

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26

Counts Against A Background

The last chapter dealt with plotting distributions of counts in the case where the counts are to be plotted "against an empty sky". In doing this, we were really using a background although we did not supply one. Its use was inevitable, and unconscious rather than explicit and helpful.

In the present chapter we deal with plotting counts against a background. Here the single most important issue is the comparison of observed counts with a description of what might have been. Here we want to see a whole array of residuals along with observations and partial description.

26A. The invisible standard of comparison

From time to time we have come back to the central issue of graphical plotting:

- Every picture generates a standard of comparison -- whether we want it to do so or not, whether we provide for it explicitly or try to prevent its use.

- Specifically, if the eye can use a straight line for the standard of comparison, it will do just that.

In view of these inescapable facts, what comparison is implicitly in mind whenever we draw any of the pictures we saw in the last chapter? The strongest standard of comparison is a horizontal line (or occasionally a vertical one). When will the revealing parts of our pictures -- in this chapter usually the ends of some kind of bar or block -- actually follow such

a line? Exactly when the counts observed for the various intervals are all the same, when we are dealing with what is often called a rectangular distribution.

Do rectangular distributions -- or even reasonable approximations to such -- occur frequently in practice? Every honest analyst of data that I know will answer this question with a "no". Why do we use such an unlikely situation as an ideal reference? Because we are foolish? Not really. Rather because anything much better is likely to:

- have to be tailored, in some detail, to the particular distribution we are examining.

A single plotting scheme serves to make all rectangular distributions reveal themselves through a variety of horizontal straight lines. There is no way to gain an equivalent generality for any other shape with a single plotting scheme. (Proof?) As a result, if we want to go further, as we do and will, we must be prepared to make and look at pictures each of which compares a single set of data with a single particular distribution. To do this will raise a variety of new problems, but such problems are only to be expected -- once we recognize the exceptional nature of the behavior of all rectangular distributions.

Rather clearly the first general sort of distribution we shall want to look at more effectively consists of more or less symmetrical and single-humped distributions. (The example we used throughout the last chapter is a reasonable example; it comes within the scope of "more or less". We

will continue to use it.) In order to adjust our picture to be more revealing about the shape of such a distribution we need to know something about both where it is centered and how broadly it is spread.

As a consequence, we will have to give some attention to assessing location and spread and using the results of assessment to select a basis for comparison. Before we do this, however, we will begin by looking at the final step, by asking:

- given a set of values, and something with which to compare them, how do we make an effective comparison?

26B. Hanging the picture

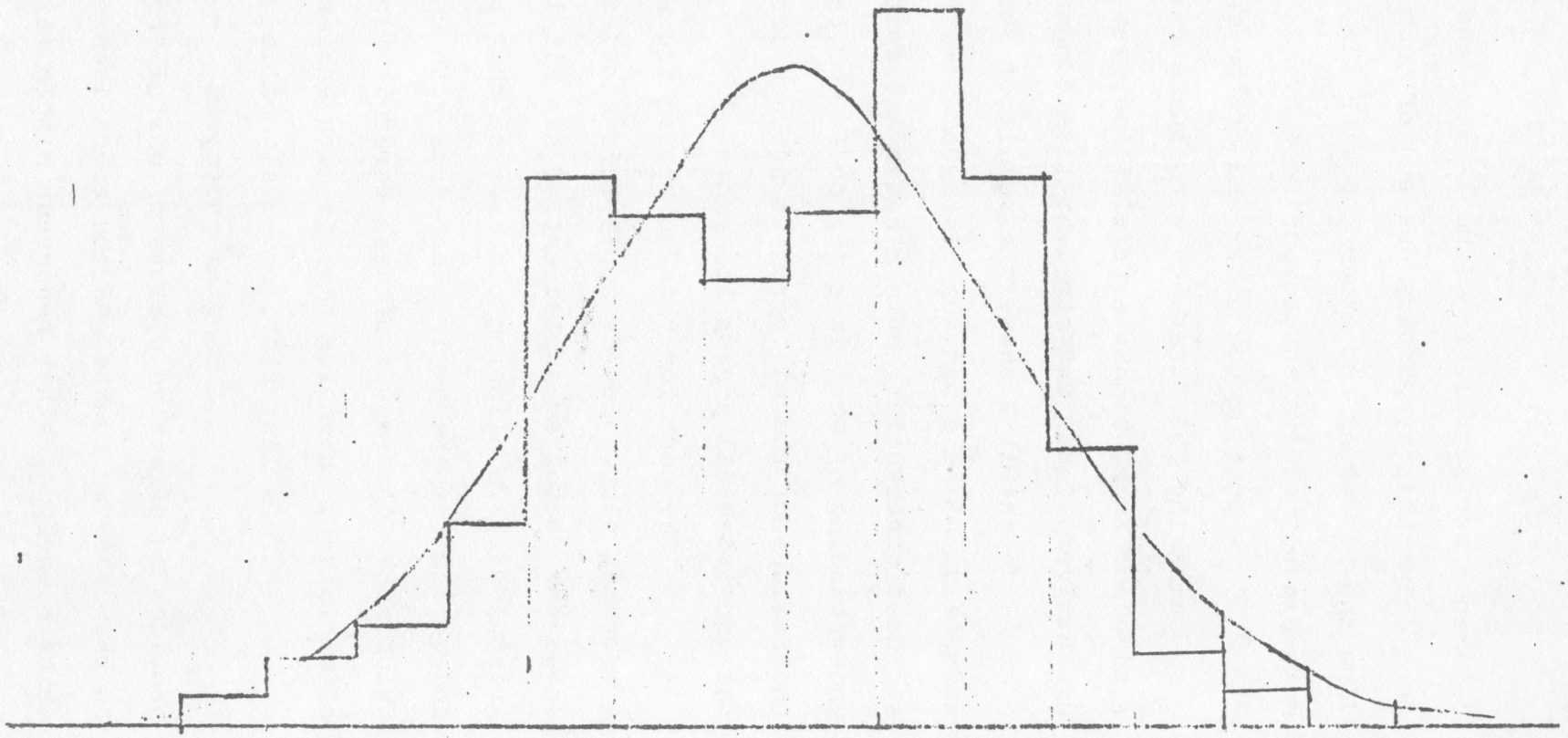
Let us continue with our batch of 120 residuals. Exhibit 1 shows the histogram and a smooth curve of distribution, about whose origin we shall learn in the next section. In both area is proportional to count. This curve assigns, as we will also learn, respectively, 19.0, 16.1, 11.7, 7.2, 3.8, 1.7 and 0.6 cases to the intervals of unit length starting out from zero in either direction.

We can sit and look very hard, trying to compare the bars of the histogram with the arch of the curve. To do this is, of course, both wasteful and unwise. Comparing with a curve -- using a curve as a standard of comparison -- is always poor graphics. We should do better.

When we stop to say what we are doing, it is easy to see how to do better. We have vertical bars of specified height, which we are comparing with the vertical separation

exhibit 1 of chapter 26 -- Nebraska residuals

Histogram (as in exhibit 5 of chapter 25) and smooth comparison curve



between a horizontal line (the x-axis) and a curve. We have been doing this by putting one end of each bar on the straight line and comparing the other end with the curve. We could make exactly equivalent comparisons by:

- putting the other end of the bar on the curve and comparing the first end with the straight line.

A simple change, almost a trivial one, yet a change that gains us that very important straight-line comparison standard.

Exhibit 2 shows the result of such a slipping, naturally called a hanging histogram. We now have a fair look at the discrepancies between batch and comparison curve. Our understanding of what is going on is, of course, hampered by the failure of the histogram to produce irregularities of about the same size all across its range.

The cure for this we know: use a rootogram -- or a rootobar plot. Let us apply this knowledge. Exhibit 3 shows the rootogram equivalent of exhibit 1. There is still great profit in slipping our bars so that they fit the curve -- and we can compare their other ends with the x-axis. Exhibit 4, which shows the result of such careful slipping, is the analog of exhibit 2, an analog that is much easier to interpret.

This last plot, obviously called a

hanging rootogram

provides an effective picture of useful "residuals" corresponding to the observed counts and the comparison curve. We now see quite clearly a rather recurrent ripple (almost as if periodic!)

exhibit 2 of chapter 26 -- Nebraska residuals

The same comparison curve with hanging histogram bars

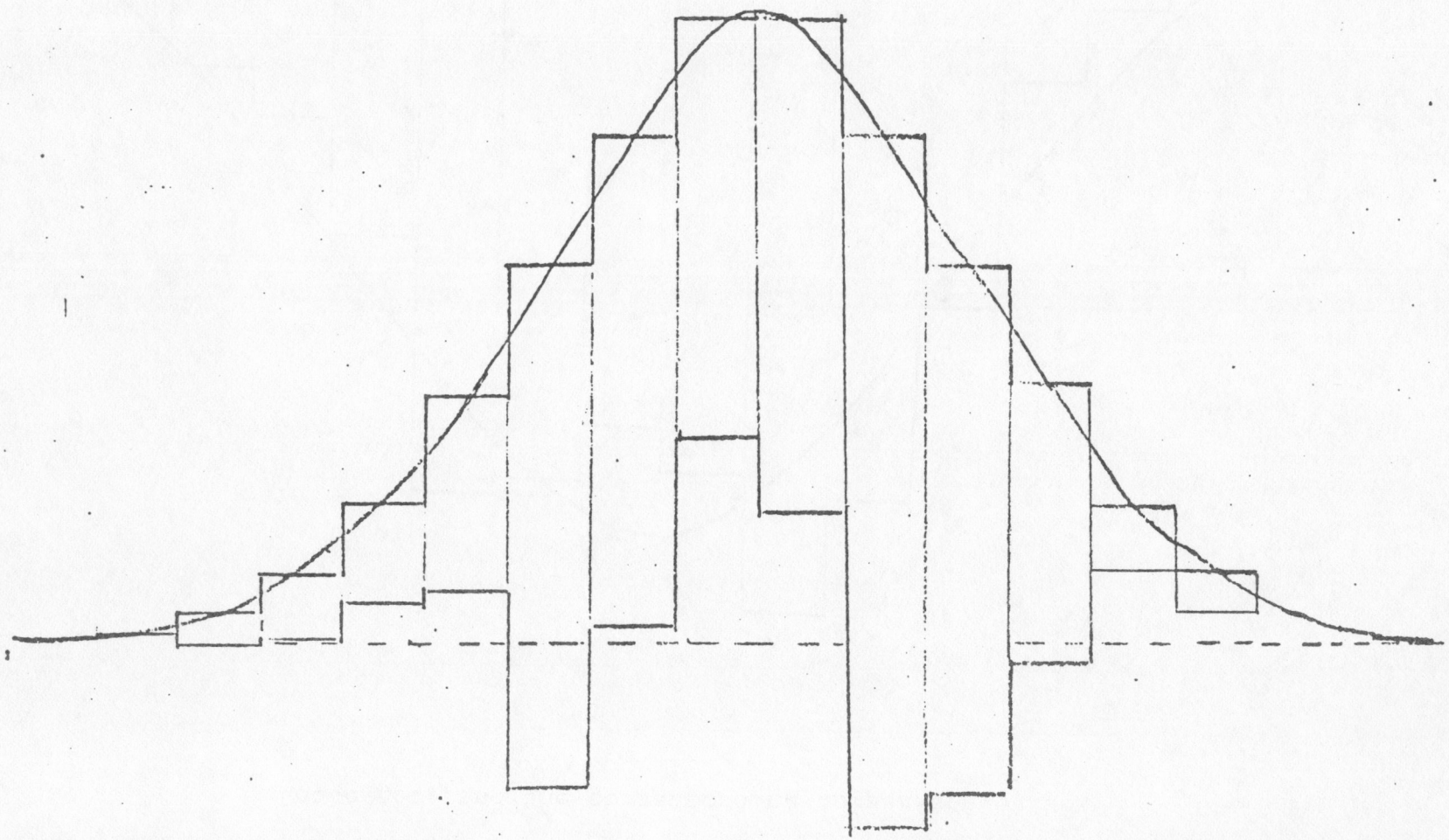


exhibit 3 of chapter 26 -- Nebraska residuals

Rootogram, and the corresponding comparison curve

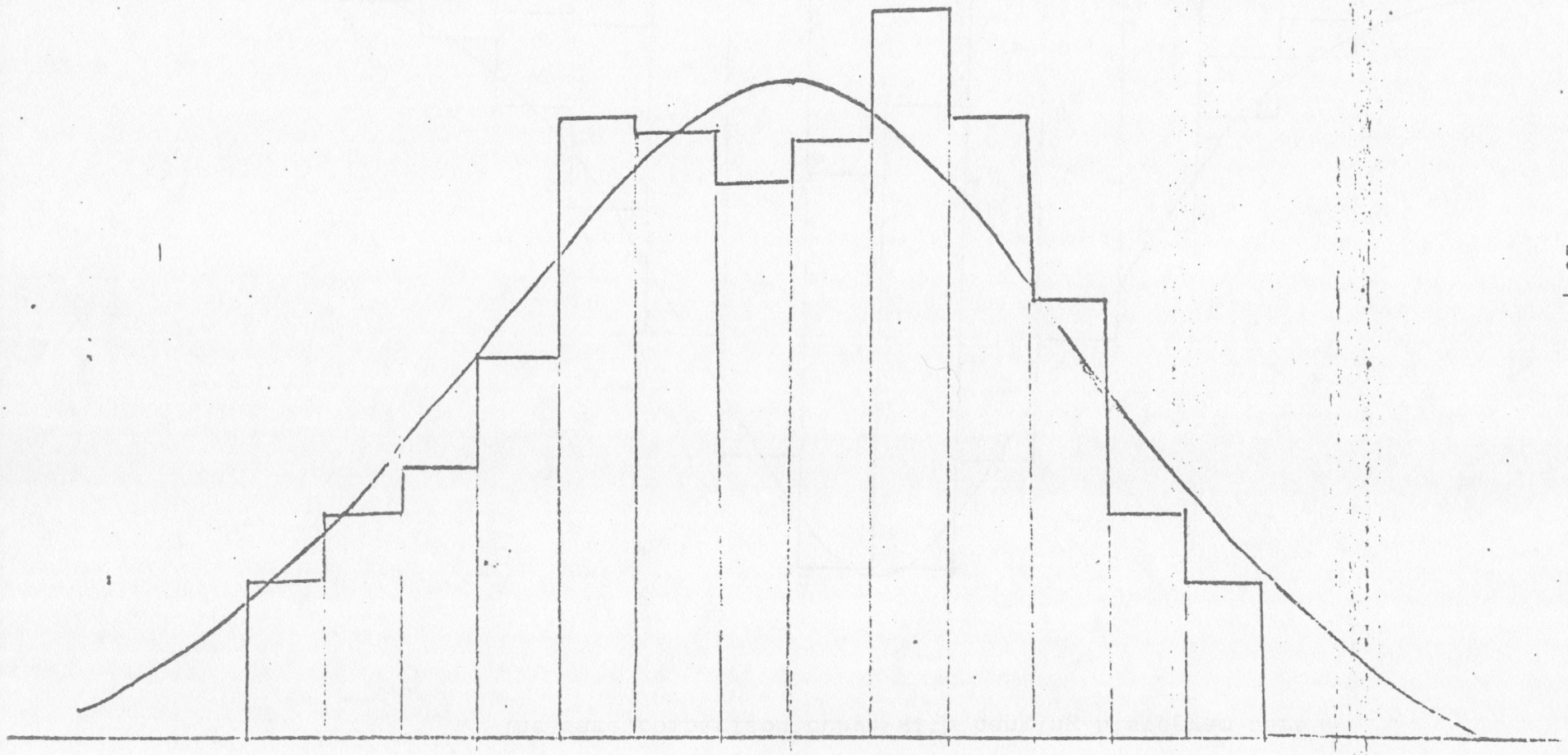
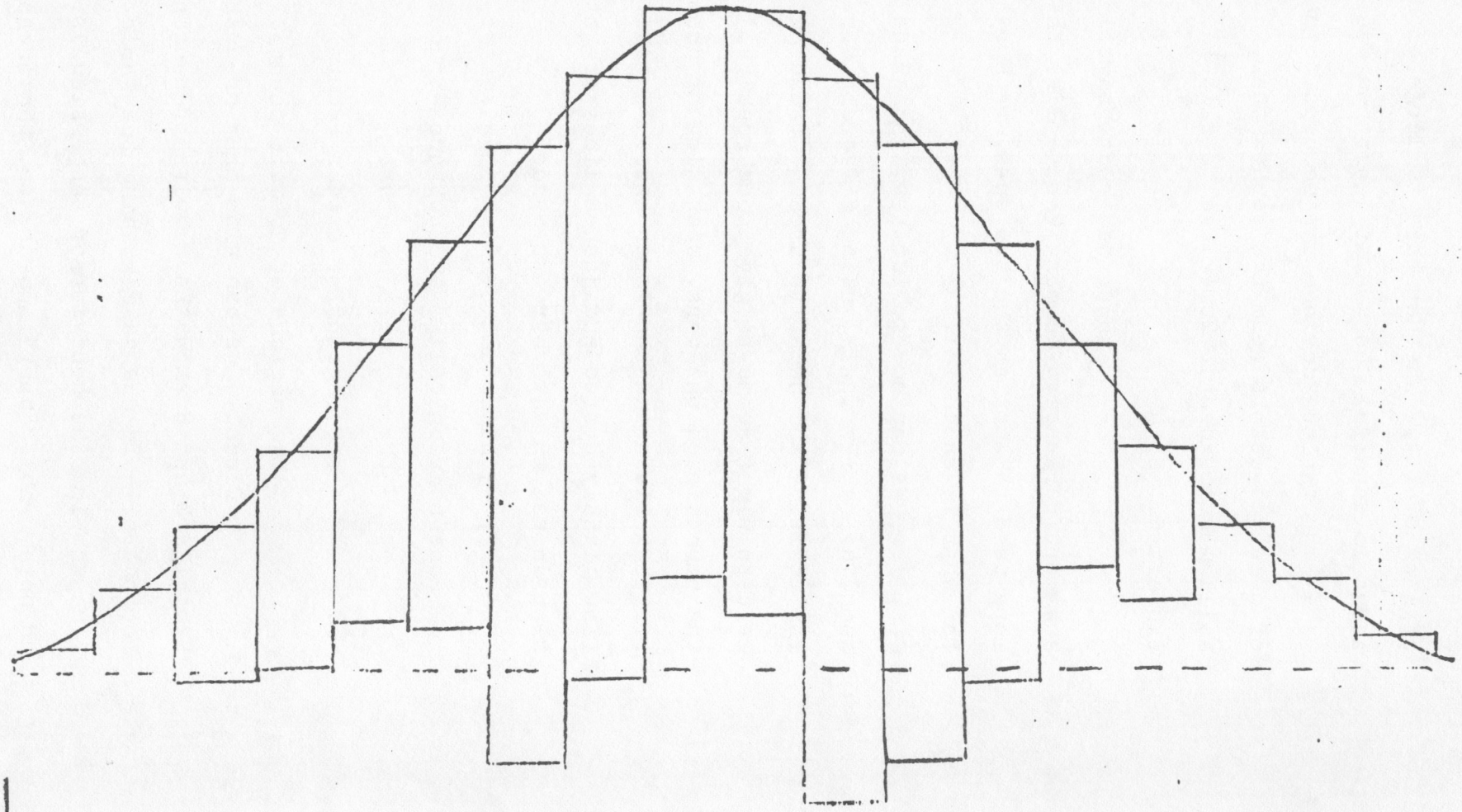


exhibit 4 of chapter 26 -- Nebraska residuals

The same comparison curve with hanging rootogram bars



of lack of fit. Whether this is large enough for us to worry about is doubtful, but we at least now see it clearly.

Various devices can be used to make the deviations of observed counts from the counts that would match the curve still more evident. Some like to flip the picture over as in exhibit 5. Such

suspended rootograms

are more persuasive to many who -- like the author -- are more used to look at the top ends, rather than the bottom ends, for what is essential.

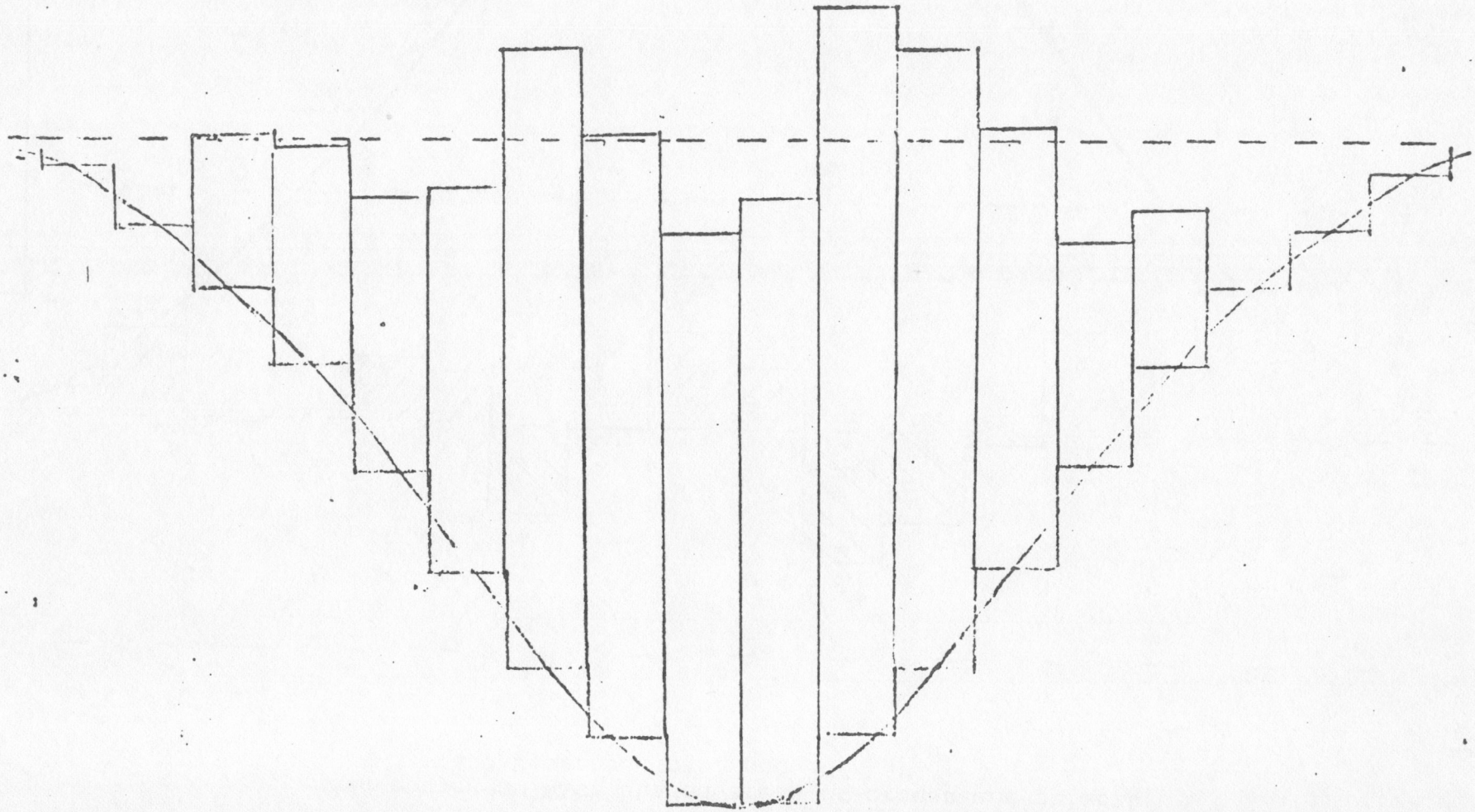
We can now go further, and shade in the areas corresponding to residuals, using different directions of shading for positive and negative areas. (Shading outwards from left to right seems to give good results.) We can also reduce the emphasis on the bars, weakening or dropping both the vertical sides and the lower ends. Exhibit 6 shows one version. (The lightly dotted and dashed lines are at ±1 and ±1.5.) This plot shows, in ways hard to evade:

- the actual counts (on a square-root scale, as shown graphically, and <u>not</u> area-true).
- the comparison curve.
- the regions of excess counts.
- the regions of deficit counts.

Clearly, we can also hang or suspend rootobars. Exhibits 7 and 8 show the results of hanging and suspending rootobars -- in one case keeping the whole of the rootogram's broken line showing the behavior of the residuals. In order to avoid an unfavorable "busy" appearance, the areas of excess and deficit in these plots have NOT been shaded.

exhibit 5 of chapter 26 -- Nebraska residuals

Exhibit 4 turned over



Suspended rootogram with shaded areas of excess and deficit
and weakened lower ends of bars

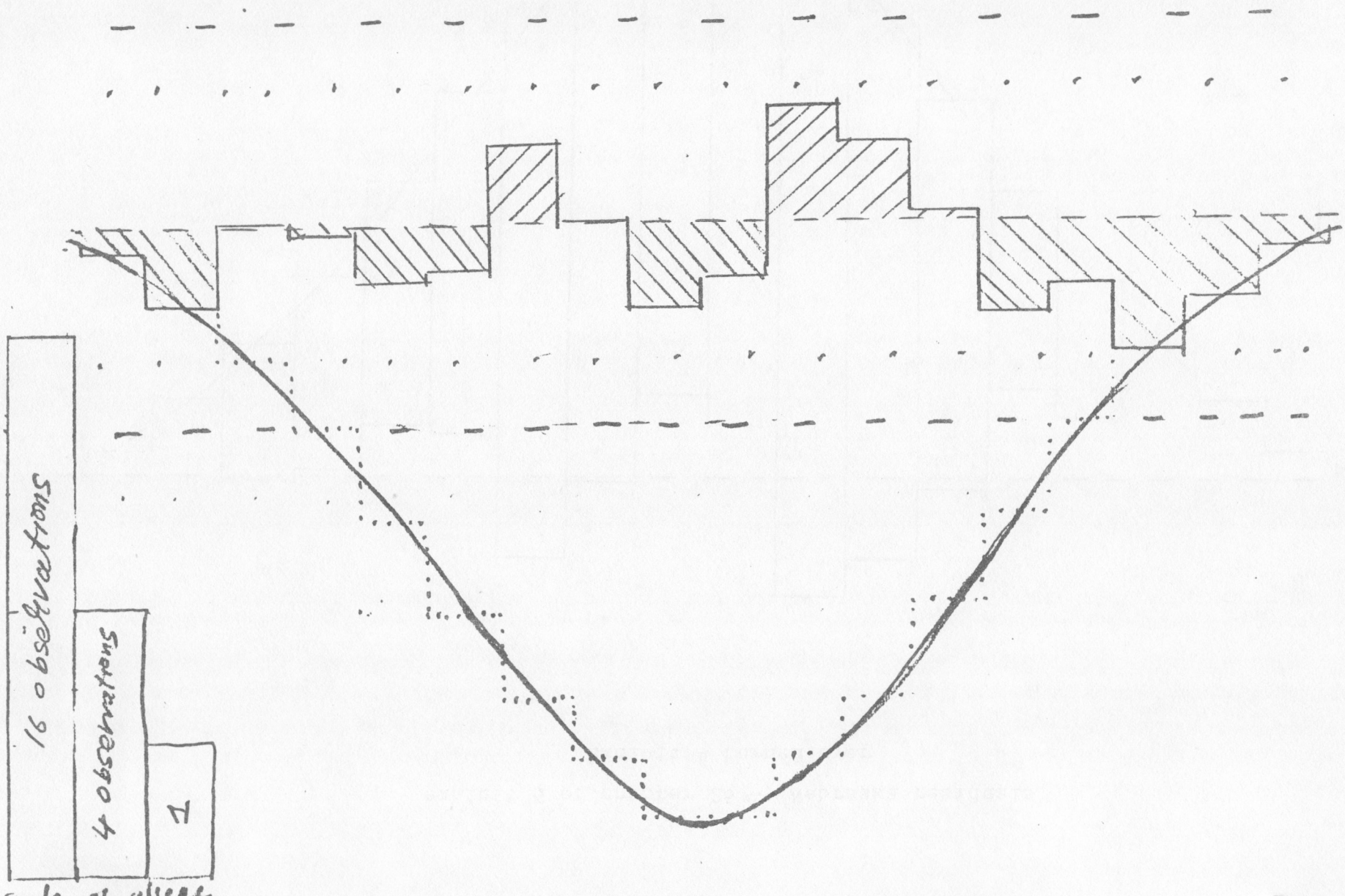
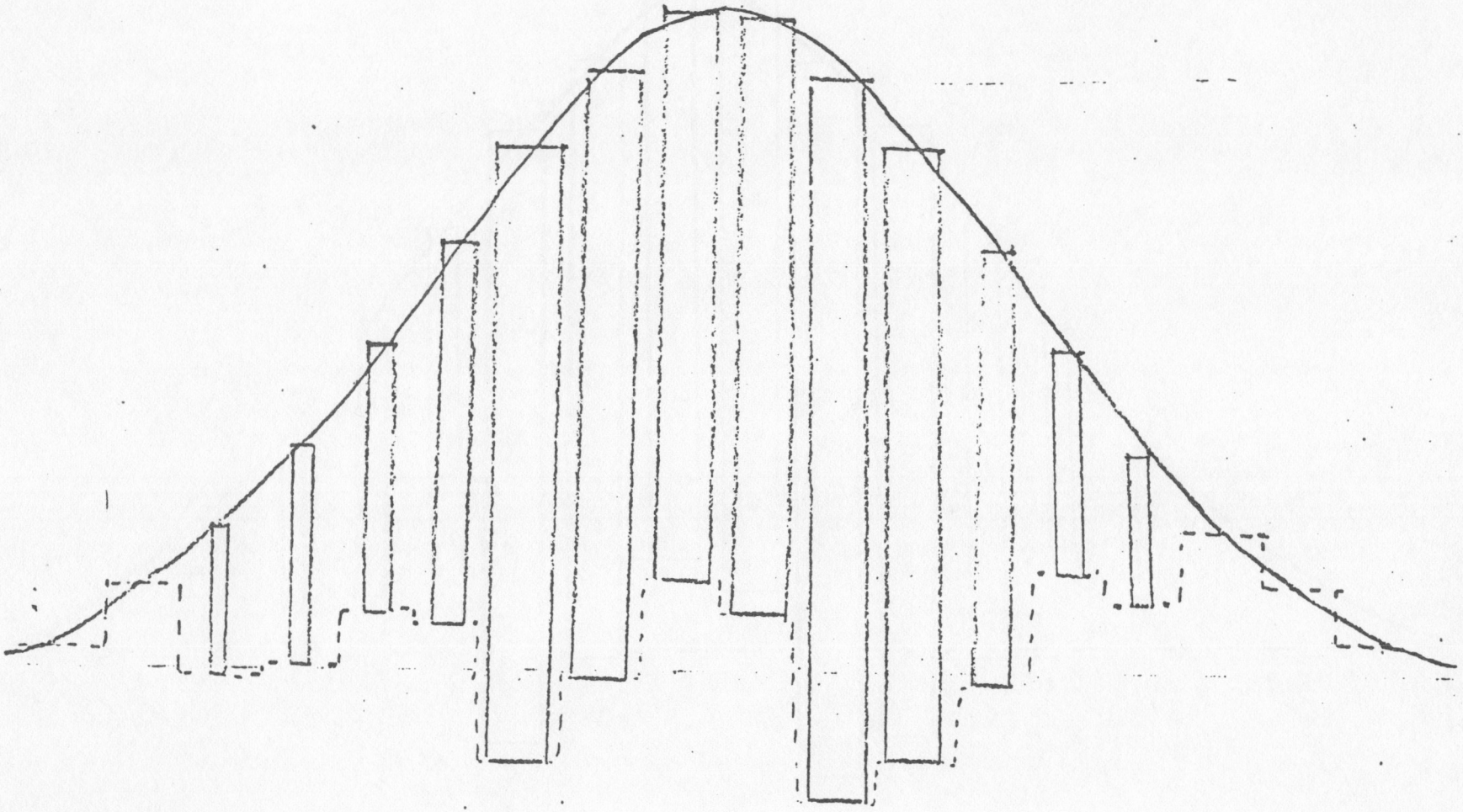


exhibit 7 of chapter 26 -- Nebraska residuals

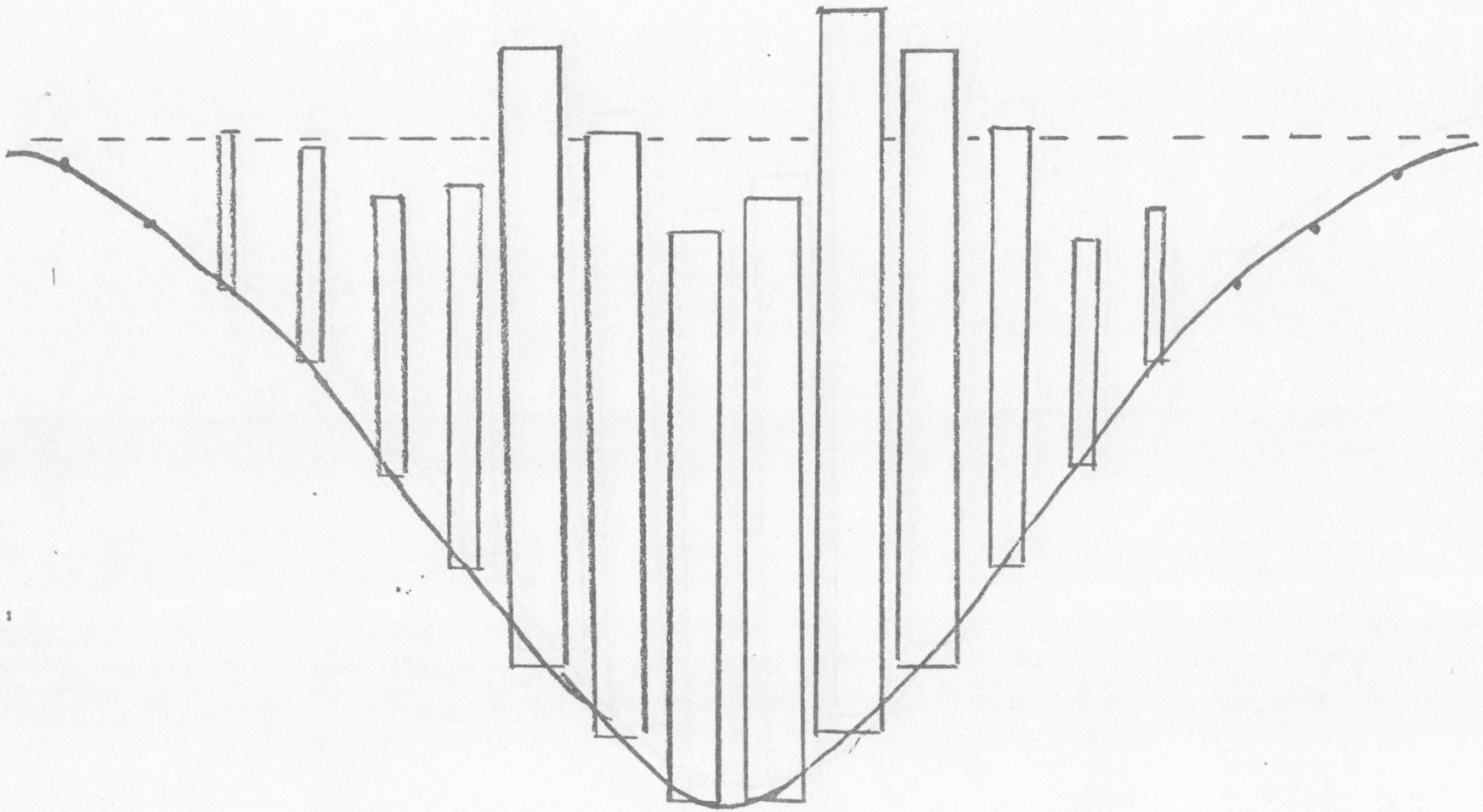


Increasing residual →

Hanging isolobars

exhibit 8 of chapter 26 -- Nebraska residuals

Suspended rootobars



26C. Three comparison curves

It would be a mistake to have only one kind of comparison curve. The world is not that unified -- histograms and rootograms of actual counts do not all look that much alike. We need to expect a choice among alternative backgrounds.

We are going to provide mechanism here and now for using three kinds of comparison curves. (As an appendix to a later chapter we will offer the possibility of using three more.) The three that we shall consider are called

rectangular

Gaussian

and

Cauchy

The first is just what its name implies. The second is often miscalled "normal" -- miscalled, since it is not usual for data to follow it all that closely. The third has longer tails.

Roughly speaking, Gaussian distributions only exist in abstract theory. Their absence from the real world does nothing to keep them from serving us reasonably well:

- as a convenient and conventional basis for comparison, to be used both with data and with theoretically defined distributions.

- as a first approximation, when we know only -- or wish to use only the knowledge of -- something about location and spread and yet require a more detailed distribution for what we are to do.

Our present activity, using comparison curves for histograms and rootograms, is a special case of each of these uses. We are involved in comparison -- and we do want to use only information about location and spread.

Thus it should be no surprise that most hanging and suspended histograms and rootograms -- at least on the first try -- are hung on Gaussian comparison curves.

The fact that we do this definitely does NOT mean that we believe that the world IS Gaussian. What we do believe is that

- comparing the world with a Gaussian background is often helpful -- as much for revealed differences as for revealed similarities.

* * *

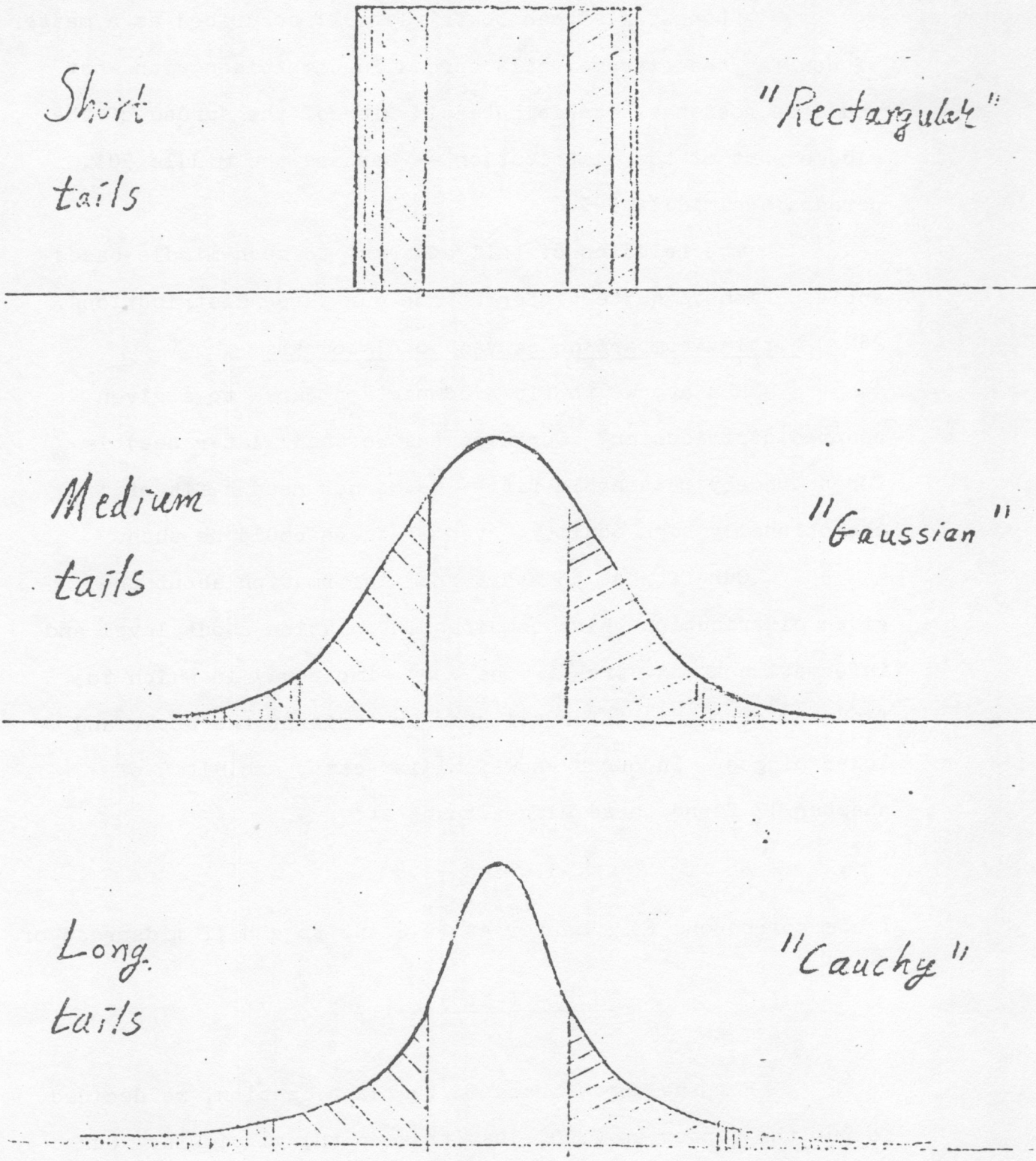
Before we go on with our example, we ought to strengthen our grasp on the three shapes of comparison curve. What are these shapes like? Exhibit 9 shows them. In it we have chosen to match among the curves:

- the spacing (and location) of the hinges (quartiles) (shown by the beginning of shading).
- the central height.

Notice carefully what it means for one shape of distribution to have "longer tails" than another. This is NOT just a matter of breadth. It is easy to think of a very broad Gaussian distribution being compared with a very concentrated Cauchy distribution -- when the latter will still be said to have longer tails.

exhibit 9 of chapter 26 -- standard shapes

The three shapes of comparison curves dealt with in exhibit 9



Note: Middle 50% unshaded, both end 10%'s doubly shaded

"Long tails" can be fairly well described as a matter of how far the extreme tails spread in comparison with what would be reasonably anticipated in view of the spread of the middle part of the distribution -- perhaps the middle 50%, perhaps the middle 90%.

The relation of tail behavior to such middle-based anticipation is quite different for our three distributions.

26D. Fitting comparison curves to histograms

How are we to fit a comparison curve to a given counted distribution? Our need (as we shall later see) is for a crudely reasonable fit -- we do not need a fit of exceptionally high quality, even if there could be such.

Our fit has to start from information about the given distribution which combines information about level and information about spread. One very simple way in which to find and give this information is in terms of the upper and lower hinges. In our by-now-familiar case, exhibit 5 of chapter 17 found these hinges to be at:

$$-1.6 \text{ and } +1.8$$

These correspond to a center at +0.1 and to a half midspread of

$$\frac{+1.8 - (-1.6)}{2} = 1.7$$

For the curves used in the last section, we decided to put the center at 0.00, just to keep things simple. Our data was collected in intervals each 1.00 long, which is

$$\frac{1.00}{1.7} = .59$$

of a half midspread. We may as well approximate .59 by .6. The centers of our intervals, when measured in half midspreads, will now lie at

$$\tfrac{1}{2}(0.0+0.6) = 0.3$$

$$\tfrac{1}{2}(0.6+1.2) = 0.9$$

and so on -- at ...,-4.5,-3.9,-3.3,-2.7,-2.1,-1.5,-0.9,-0.3, +0.3,+0.9,+1.5,+2.1,+2.7,+3.3,+3.9,... .

* to histograms *

Our next problem is to calculate how many counts the comparison curve assigns to each of our intervals -- in particular, say, to the interval from +0.6 to +1.2. For graphical accuracy, it will usually be precise enough to take this count, given by the area under this part of the curve, as proportional to the height of the curve at the center of the interval. (For our selected interval, the height at +0.9.) Thus we need:

- to find the relative heights of the curve at the midpoints of our intervals.
- to scale up -- or down -- these heights by a factor chosen to make the total count equal to -- or close to -- the 120 which is the observed count.

A simple basis for these calculations is exhibit **10**, which gives heights -- and, well out in the tail, the remaining fractions for the three comparison curves, something we may learn later to use.

exhibit 10 of chapter 26 -- table

Relative HISTOGRAM heights for three comparison curves

(Each scaled to put the hinges at ±1.0)

	Relative height		
Scale	Rect.	Gauss	Cauchy
0.0	1.00	1.00	1.00
±0.1	1.00	1.00	.99
±0.2	1.00	.99	.96
±0.3	1.00	.98	.92
±0.4	1.00	.96	.86
±0.5	1.00	.94	.80
±0.6	1.00	.92	.74
±0.7	1.00	.89	.67
±0.8	1.00	.86	.61
±0.9	1.00	.83	.55
±1.0	1.00	.80	.50
±1.1	1.00	.76	.45
±1.2	1.00	.72	.41
±1.3	1.00	.68	.37
±1.4	1.00	.64	.34
±1.5	1.00	.60	.31
±1.6	1.00	.56	.28
±1.7	1.00	.52	.26
±1.8	1.00	.48	.24
±1.9	1.00	.44	.22
±2.0	falls	.40	.200
±2.1	0.00	.37	.185
±2.2	.00	.33	.171
±2.3	.00	.30	.159
±2.4	.00	.27	.148
±2.5	.00	.242	.138
±2.6	.00	.214	.129
±2.7	.00	.190	.121
±2.8	.00	.167	.113
±2.9	.00	.147	.106
±3.0	.00	.129	.100

	Relative height		
Scale	Rect. (*)	Gauss (*)	Cauchy (*)
3.0	.00	.131	.100
	(.000)	(.022)	(.102)
3.2	.00	.100	.089
	(.000)	(.016)	(.097)
3.4	.00	.074	.080
	(.000)	(.011)	(.091)
3.6	.00	.054	.072
	(.000)	(.008)	(.086)
3.8	.00	.038	.065
	(.000)	(.005)	(.082)
4.0	.00	.027	.059
	(.000)	(.004)	(.078)
4.2	.00	.0185	.054
	(.000)	(.002)	(.074)
4.4	.00	.0125	.049
	(.000)	(.002)	(.071)
4.6	.00	.0084	.045
	(.000)	(.001)	(.068)
4.8	.00	.0054	.042
	(.000)	(.001)	(.066)
5.0	.00	.0035	.038
	(.000)	(.000)	(.063)
6.0	.00	.0002	.027
	(.000)	(.000)	(.052)
7.0	.00	.0000	.020
	(.000)	(.000)	(.045)
8.0	.00	.0000	.0154
	(.000)	(.000)	(.040)
9.0	.00	.0000	.0122
	(.000)	(.000)	(.036)
10.0	.00	.0000	.0099
		(.000)	(.034)

(*) These entries give the fraction of area in a single tail beyond the scale value for the line immediately above -- the one-tail area.

Turning to our example, we need to carry out the calculations set out in exhibit 11. We have a total of 120 counts. The relative heights found from exhibit 10 add up to 6.186. The fitted counts should then be

$$\frac{120}{6.186} = 19.4$$

times as large as the tabled counts.

If we halve the intervals for positive values, so that the widths are no longer all alike, we have to add one step to the calculation -- the step already required to find observed counts per unit base. Exhibit 12 shows the calculations.

* to rootograms *

One way to fit comparison curves to rootograms would be to use the counts to find the comparison curve for the corresponding histogram -- and then to take square roots of both counts and histogram curves. This has certain aspects of neatness and order -- but involves extra calculation.

The fast way to reach a comparison curve -- one whose quality is almost always satisfactory -- is to have a table of relative heights of rootogram curves and fit them to the values of

$$\sqrt{\text{counts per unit base}}$$

through which the observations must appear. Exhibit 13 provides the necessary table. Exhibit 14 tackles the counts and equal-

exhibit 11 of chapter 26 -- Nebraska residuals

Fitting a Gaussian comparison curve to the histogram corresponding to the data of exhibit 1 of chapter 25

A) FIXING THE CONVERSION

hinges:	-1.6 and +1.8	in base values
midspread:	3.4 = 1.8 - (-1.6)	
2/midspread:	.59 = 2/3.4	.6 accepted
.6·hinges:	-0.96 and +1.08	
center:	+0.06	.0 accepted

Conversion: .6(base value) + 0.0

B) USING THE CONVERSION AND EXHIBIT 10

Center values		Gaussian counts		Actual counts	
Base	**Converted**	**Tabled**	**Fitted**	**Obsd**	**Residual**
-6.5	-3.9	.032	.6	1	.4
-5.5	-3.3	.087	1.7	2	.3
-4.5	-2.7	.194	3.8	3	-.8
-3.5	-2.1	.37	7.2	6	-1.2
-2.5	-1.5	.60	11.7	16	4.3
-1.5	-0.9	.83	16.1	15	-1.1
-0.5	-0.3	.98	19.0	13	-6.0
0.5	0.3	.98	19.0	15	-4.0
1.5	0.9	.83	16.1	22	-5.9
2.5	1.5	.60	11.7	16	4.3
3.5	2.1	.37	7.2	8	.8
4.5	2.7	.194	3.8	2	-1.8
5.5	3.3	.087	1.7	1	-.7
6.5	3.9	.032	.6	0	-.6
(sums)		(6.186)	(120.2)	(120)	

$$\frac{\text{Fitted}}{\text{Tabled}} = \frac{120}{6.186} = 19.4$$

likely {left-to-right / down-to-up} coordinates in plots.

C) NOTE

Slide-rule accuracy will almost always be sufficient when changing from tabled to fitted values. Use of a slide rule here requires only one setting and is very convenient. (The setting can be made by the division of one sum by the other.)

D) PROBLEM

Make the calculation, and the plot for a Cauchy comparison curve.

exhibit 12 of chapter 26 -- Nebraska residuals

Calculation of a Gaussian comparison curve for a histogram with unequal intervals

A) FINDING CONVERSION -- as in **A** of exhibit 11

B) USING CONVERSION -- ratio = count/width

Interval (converted)		Observed		Gaussian ratios		Residuals
Center	**Width**	**Count**	**Ratio**	**Tabled**	**Fitted**	
-3.9	.6	1	1.7	.032	1.0	.7
-3.3	.6	2	3.3	.087	2.9	.4
-2.7	.6	3	5.0	.194	6.4	-1.4
-2.1	.6	6	10.0	.37	12.1	-2.1
-1.5	.6	16	26.7	.60	19.7	7.0
-.9	.6	15	25.0	.83	27.2	-2.2
-.3	.6	13	21.7	.98	32.1	-10.4
.15	.3	8	26.7	.99	32.4	-5.7
.45	.3	7	23.3	.95	31.1	-7.8
.75	.3	15	50.0	.88	28.8	21.2
1.05	.3	7	23.3	.78	25.6	-2.3
1.35	.3	12	40.0	.66	21.6	18.4
1.65	.3	4	13.3	.55	18.0	-4.7
1.95	.3	5	16.7	.42	13.7	3.0
2.25	.3	3	10.0	.32	10.5	-.5
2.55	.3	2	6.7	.21	6.9	-.2
2.85	.3	0	0.0	.160	5.2	-5.2
3.15	.3	1	3.3	.107	3.5	-.2
3.45	.3	0	0.0	.069	2.3	-2.3
3.75	.3	0	0.0	.042	1.4	-1.4
(sums)			(301.7)	(9.231)		

$$\frac{\text{Fitted}}{\text{Tabled}} = \frac{301.7}{9.231} = 32.7$$

Examples: 1.7 = 1/0.6, 3.3 = 2/0.6,...
1.032 from -3.9 through exhibit 10, .087 from -3.3,...
1.0 = .032x32.7, 2.9 = .087x32.7,...
.7 = 1.7-1.0, .4 = 3.3-2.9,...

= likely left-to-right coordinate
= likely down-to-up coordinates
} in plots

Remark: The slide-rule note of exhibit 11 applies equally here.

C) PROBLEM

Do the calculations -- and make the plot -- for a Cauchy comparison curve.

exhibit 13 of chapter 26 -- table

Relative ROOTOGRAM heights for three comparison curves
(Each scaled in hinge units -- hinges at ±1.0)

A) TABLE

Scale	Relative height	
	Gauss	Cauchy
±0.0	1.00	1.00
±0.1	1.00	.99h
±0.2	.99h	.98
±0.3	.99	.96
±0.4	.98h	.93h
±0.5	.97	.89h
±0.6	.95h	.86
±0.7	.94	.82
±0.8	.92h	.78
±0.9	.91	.74
±1.0	.89	.70h
±1.1	.87	.67
±1.2	.84h	.64
±1.3	.82	.61
±1.4	.79h	.58h
±1.5	.77	.55h
±1.6	.74h	.53
±1.7	.72	.51
±1.8	.69	.49
±1.9	.66	.47
±2.0	.63	.45
±2.1	.60	.43
±2.2	.57	.41
±2.3	.54h	.40
±2.4	.52	.38h
±2.5	.49	.37
±2.6	.46	.37
±2.7	.43h	.35
±2.8	.41	.33h
±2.9	.38	.32h
±3.0	.36	.31h

Scale	Relative height	
	Gauss	Cauchy
±3.0	.36	.31h
±3.1	.34h	.30h
±3.2	.31	.30
±3.3	.29	.29
±3.4	.27	.28
±3.5	.25	.27h
±3.6	.23	.27
±3.7	.21	.26
±3.8	.19h	.25h
±3.9	.17h	.25
±4.0	.16	.24
±4.1	.15	.23h
±4.2	.13h	.23
±4.3	.12	.22h
±4.4	.11	.22
±4.5	.10	.21h
±4.6	.09	.21
±4.7	.08	.21
±4.8	.07	.20h
±4.9	.06h	.20
±5.0	.06	.19h
±5.2	.04h	.19
±5.4	.03h	.18
±5.6	.03	.17h
±5.8	.02	.17
±6.0	.01h	.16h
±6.5	.01	.15
±7.0	.00h	.14
±7.5	.00	.13
±8.0	.00	.12h
±9.0	.00	.11
±10.0	.00	.10
±11.0	.00	.09
±12.0	.00	.08
±13.0	.00	.07h
±14.0	.00	.07

"h" means "and a half", thus ".99h" means ".995".

(Cont'd)

exhibit 13 (Cont'd)

B) REMARKS

For most purposes, interpolation in this table can -- and should -- be in a light-hearted vein. For example:

Given	**Thought process** (for Gauss column)
2.13	Less than half-way from .60 to .57, let's say .59
-1.37	More than half-way from .82 to .79h, perhaps .80h or .80. .80 is simpler, let's say it.
-.95	Half-way from .91 to .89, so let's say .90.
+2.05	Half-way from .63 to .60, let's say .61h -- unless you want .62.

C) PROBLEM

Write down appropriate thought processes starting with -2.45, +1.05, -1.72, +.48, and -4.83.

exhibit 14 of chapter 26 -- Nebraska residuals

Fitting a rootogram comparison curve to the counts and equal-width intervals of exhibit 11

A) FIXING THE CONVERSION

As in exhibit 11.

B) USING THE CONVERSION AND EXHIBIT 13 -- root = $\sqrt{\text{count}}$

Center values		Observed		Gaussian roots		Residuals
Base	**Converted**	**Counts**	**Roots**	**Tabled**	**Fitted**	
-6.5	-3.9	1	1.00	.17h	.75	.2
-5.5	-3.3	2	1.41	.29	1.25	.2
-4.5	-2.7	3	1.73	.43h	1.88	-.2
-3.5	-2.1	6	2.45	.60	2.58	-.1
-2.5	-1.5	16	4.00	.77	3.32	.7
-1.5	-0.9	15	3.88	.91	3.82	.1
-.5	-0.3	13	3.60	.99	4.27	-.7
.5	0.3	15	3.88	.99	4.27	-.4
1.5	0.9	22	4.68	.91	3.82	.9
2.5	1.5	16	4.00	.77	3.32	.7
3.5	2.1	8	2.83	.60	2.58	.2
4.5	2.7	2	1.41	.43h	1.88	-.5
5.5	3.3	1	1.00	.29	1.25	-.2
6.5	3.9	0	0.00	.17h	.75	-.8
(sums)			(35.87)	(8.34)		

$$\frac{\text{Fitted}}{\text{Tabled}} = \frac{35.87}{8.34} = 4.31$$

Examples: $-3.9 = .6(-6.5) + 0.0$, $-3.3 = .6(-5.5) + 0.0$,...
$1.00 = \sqrt{1}$, $1.41 = \sqrt{2}$,... (see note)
.17h from -3.9 through exhibit 13, .29 from -3.3,...
$.75 = (.17h)4.31$, $1.25 = (.29)4.31$,... (see note)
$.2 = 1.00-.75$, $.2 = 1.41-1.25$,... (see note)

(Cont'd)

exhibit 14 (Cont'd)

C) NOTE

There is no need to keep square roots to better than tenths in any such computation. This exhibit was done on a slide rule where hundredths came naturally. They are included for illustration, not emulation.

D) PROBLEMS

1) Repeat these calculations, keeping only tenths in the square roots.

2) Calculate a Cauchy comparison curve to fit the same data, and plot the corresponding suspended histogram.

length intervals used in exhibit 11. The only computation not equivalent to that needed for a histogram comes in calculating the square roots of the observed counts.

Exhibit 15 tackles the counts and unequal-length intervals used in exhibit 12. Here the only computation not equivalent to that needed for a histogram is almost trivial: Finding count/width is replaced by finding $\sqrt{\text{count/width}}$ -- either is a single operation on a slide rule.

26E. Polishing up suspended rootograms

Once anyone has made a fit, whether it involves one of our comparison curves or not, it is easy to find a suspended rootogram. All that is left is to find the square roots and make the plot.

In a 1903 paper, Pearson and Lee said: "whereas the eye alone cannot possibly judge whether or not his distributions follow the normal law." (Footnote to page 367.) Since we feel differently, it is worthwhile to make a suspended rootogram for the table in question. Exhibit 16 has the arithmetic. Exhibit 17 shows the suspended rootogram. We see that, except possibly for the rightmost class, the fit is excellent. If we wanted to say just how good, including or excluding these 4.5 observations, we would have to turn to numbers.

The suspended rootogram, however, lets us see all the important issues at once.

exhibit 15 of chapter 26 -- Nebraska residuals

Calculation of a Gaussian comparison curve for a rootogram
with unequal intervals
(compare exhibits 12 and 14)

A) FINDING CONVERSION -- as in **A** of exhibit 11

B) USING CONVERSION AND EXHIBIT 13 -- root = $\sqrt{\text{count/width}}$

Interval (converted)		Observed		Gaussian roots		Residuals
Center	**Width**	**Count**	**Root**	**Tabled**	**Fitted**	
-3.9	.6	1	1.3	.17h	.9	.4
-3.3	.6	2	1.8	.29	1.6	.2
-2.7	.6	3	2.2	.43h	2.4	-.2
-2.1	.6	6	3.2	.60	3.3	-.1
-1.5	.6	16	5.2	.77	4.2	1.0
-.9	.6	15	5.0	.91	4.9	.1
-.3	.6	13	4.7	.99	5.4	-.7
.15	.3	8	5.2	.99h	5.4	-.2
.45	.3	7	4.8	.98	5.3	-.5
.75	.3	15	7.1	.93	5.0	2.1
1.05	.3	7	4.8	.88	4.8	0
1.35	.3	12	6.3	.81	4.4	1.9
1.65	.3	4	3.7	.73	4.0	-.3
1.95	.3	5	4.1	.64h	3.5	.6
2.25	.3	3	3.2	.56	3.0	.2
2.55	.3	2	2.6	.47h	2.6	.0
2.85	.3	0	0.0	.39h	2.1	-2.1
3.15	.3	1	1.8	.33	1.8	.0
3.45	.3	0	0.0	.26	1.4	-1.4
3.75	.3	0	0.0	.20	1.1	-1.1
(sums)			(67.0)	(12.38)		

$$\frac{\text{Fitted}}{\text{Tabled}} = \frac{67.0}{12.38} = 5.42$$

Examples: $1.3 = \sqrt{1/.6}$, $1.8 = \sqrt{21.6}$,...
.17h from -3.9 through exhibit 13, .29 from -3.3,...
.9 = .17hx5.42, 1.6 = .29x5.42,...
.4 = 1.3-.9, .2 = 1.8-1.6,...

likely {left-to-right / down-to-up} coordinates in plots

(Cont'd)

exhibit 15 of chapter 26

C) PROBLEM

Repeat the calculation for this data and a Cauchy comparison curve.

D) NOTE

The root column is easily found on a slide rule. For example: Set 1 on B against 0.6 on A, read 1.3 on C against 1 on D.

exhibit 16 of chapter 26 -- fathers' forearm

Rootogram calculations for data and fit by Pearson and Lee to length of forearm in fathers

A) DATA, FIT, AND ROOTS

Lengths of forearm	Count of fathers		Roots		
	Obs'd	**Fitted**	**Obs'd**	**Fitted**	**Diff**
15 -15.5	1	1.9	1.0	1.4	-.4
15.5-16	6.5	6.7	2.5	2.6	-.1
16 -16.5	17	23.0	4.1	4.8	-.7
16.5-17	49	59.6	7.0	7.7	-.7
17 -17.5	125.5	119.0	11.2	10.9	.3
17.5-18	200	182.2	14.1	13.5	.6
18 -18.5	235.5	214.8	15.3	14.7	.6
18.5-19	183.5	194.3	13.5	13.9	-.4
19 -19.5	127	135.0	11.3	11.6	-.3
19.5-20	57.5	72.0	8.3	8.5	-.2
20 -20.5	31.5	29.5	5.6	5.4	.2
20.5-21	8	9.3	2.8	3.0	-.2
21 -21.5	3.5	2.2	1.9	1.5	.4
21.5-22	2 } 4.5	.5	2.1	0.7	1.4
22 -22.5	2.5				
(sum)					

B) PROBLEM

Use exhibit 13 to fit a comparison curve to the roots directly and plot the corresponding suspended rootogram.

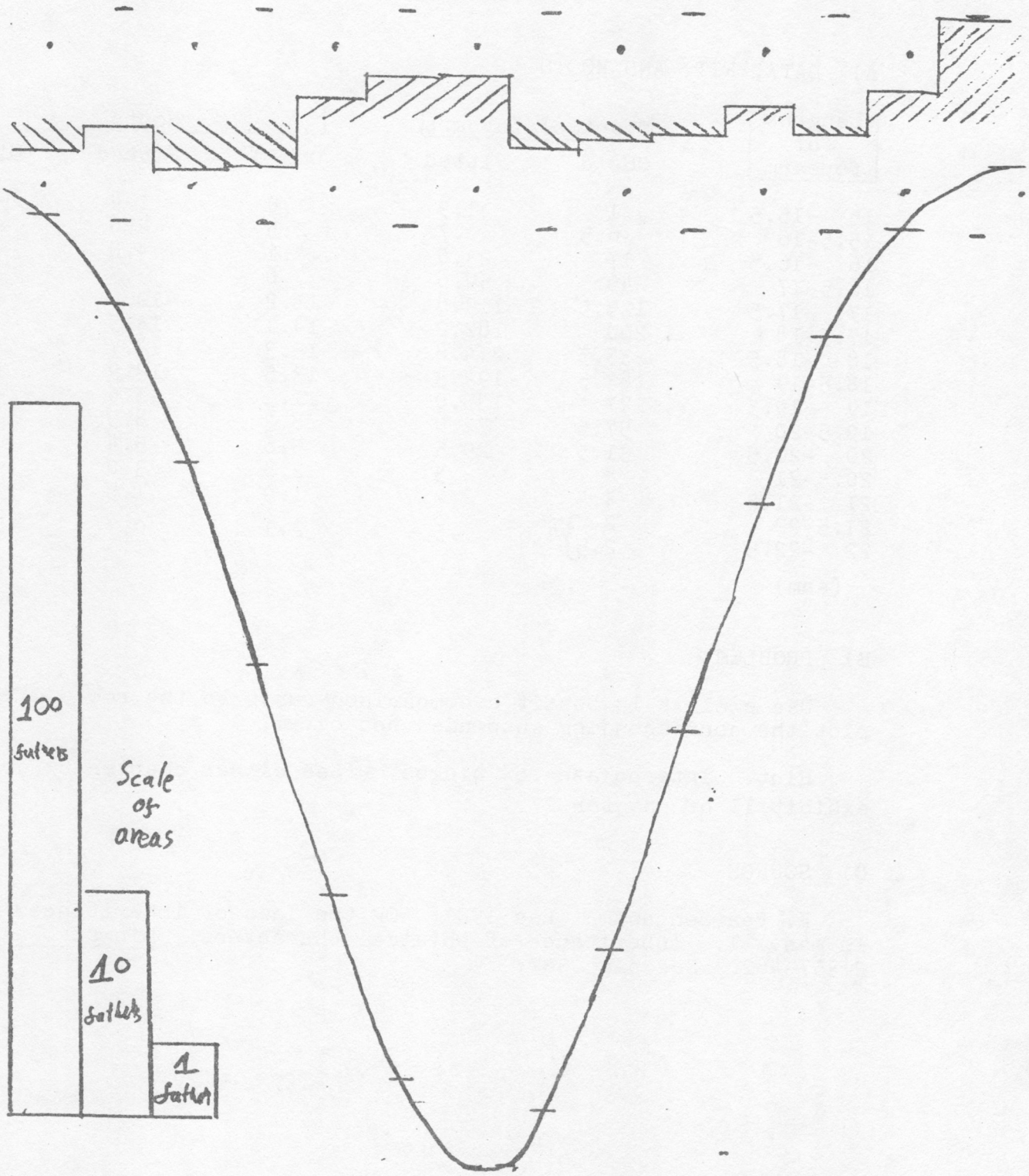
Hint. Interpolate for hinges. (See either chapter 6x or exhibit 11 of chapter 27.)

C) SOURCE

K. Pearson and A. Lee 1903. On the laws of inheritance in man. I. Inheritance of physical characters. Biometrika 2:357-462. See page 367.

exhibit 17 of chapter 26 -- fathers' forearm

Suspended rootogram for Pearson and Lee's fit
to length of forearm in fathers



Note the particular form of weakening the lower ends of the -- almost invisible -- rootogram rectangles that we used in this example.

* comparison lines *

As we gain experience with rootograms, we learn that square roots have done more for us than we realized. One reason for choosing them was their tendency to make "accidental" variations in counts appear of similar size within one rootogram. As we look at more and more instances of what appear to be "good fits" we will find that using square roots has made "accidental" variations appear to be about the same size from one rootogram to another.

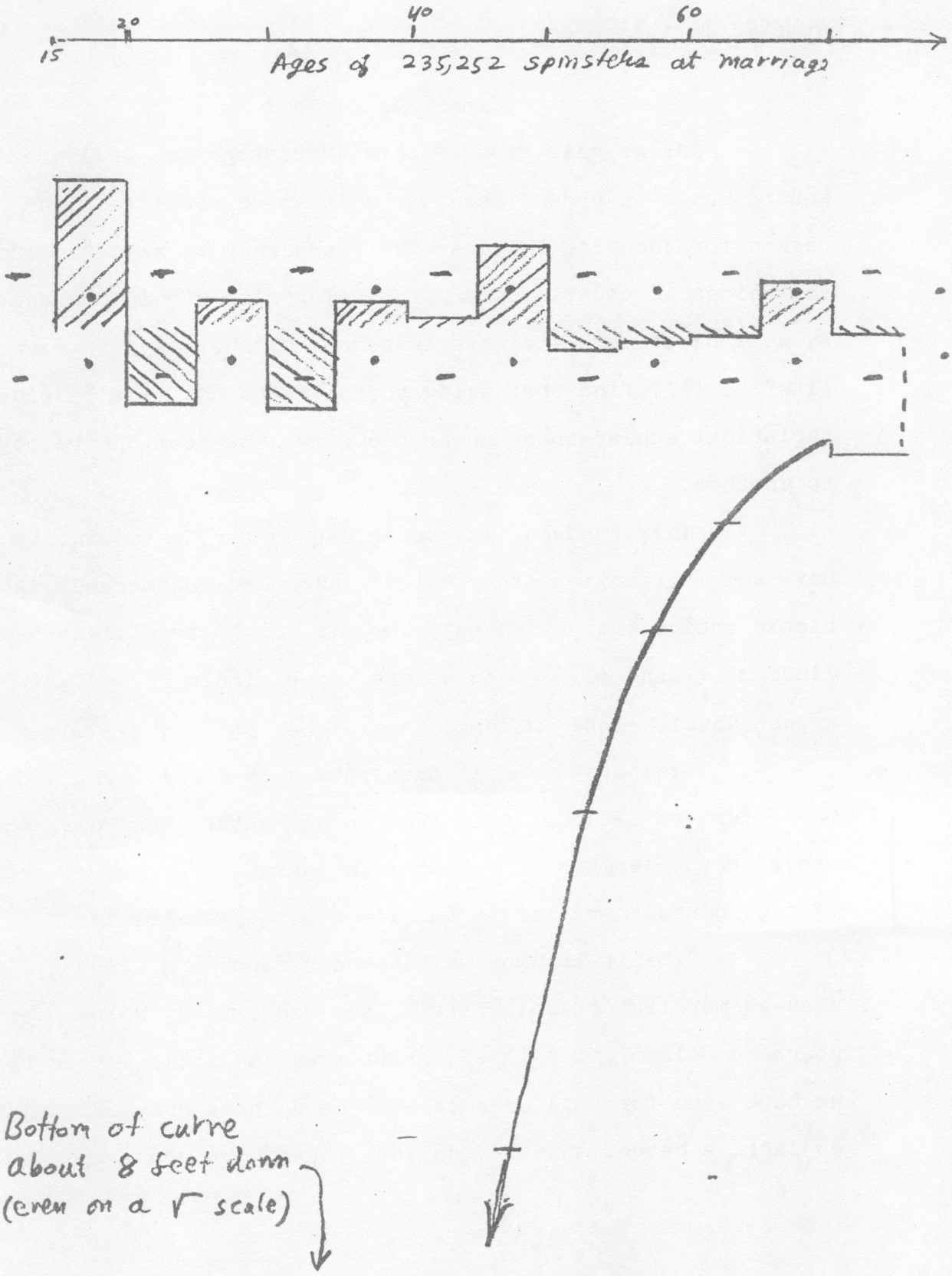
This tendency is strong enough for us to want to have some comparison lines on our suspended rootograms. A simple choice that seems quite helpful is to draw light dashed lines at ±1 and ±1.5 (when we are using simple roots). (When we use double roots, we draw these lines at ±2 and ±3.)

Exhibits 6 and 17 both show such lines lightly dotted. One refers to 120 residuals from an elementary analysis, the other to 950 lengths of father's forearms. The general relation of "accidental" scatter to the lines is quite similar.

Exhibit 18 shows a suspended rootogram for 235,252 ages at marriage of spinsters. (See exhibit 20, below, for source.) Clearly the irregularities are a little larger than we have seen for smaller batches. We do have about 250 times as large a batch, however, so that the square roots are about

exhibit 18 of chapter 26

Suspended rootogram for a large batch -- marriage ages



15 times the size of those we have looked at before. That the irregularities are about double what we have seen before is then a relatively mild increase. The constancy of size of "accidental" irregularity then appears only just a little rough.

27F. Comparing fits

The naive way to try to compare two fits would be plot separate suspended rootograms for them. Since Isserlis gave two fits to his distribution, we can try this by plotting the suspended rootogram for the other fit. Exhibit 19 gives this plot. Comparing exhibits 18 and 19 is not too difficult, so far as the most obvious matters go. We can see, for example, that the fit to the 15-20 year class is notably poorer in exhibit 19.

If we want to go a little further, it will help to do a little arithmetic to prepare for a plot that tells us more directly about the comparison. Exhibit 20 gives the arithmetic. Exhibit 21 gives a very helpful plot, in which we plot

$$2\sqrt{\text{observed}} - \sqrt{\text{fitted A}} - \sqrt{\text{fitted B}}$$

against

$$\sqrt{\text{fitted A}} - \sqrt{\text{fitted B}}$$

We look at this plot, and ask about the relation of the points to the two lines with unit slope. In this particular case, we see that the "15-", "20-" and "30-" age classes are not being well fitted by either fit -- though A

exhibit 19 of chapter 26

The same data seen from another fit -- marriage ages
(see exhibit 18)

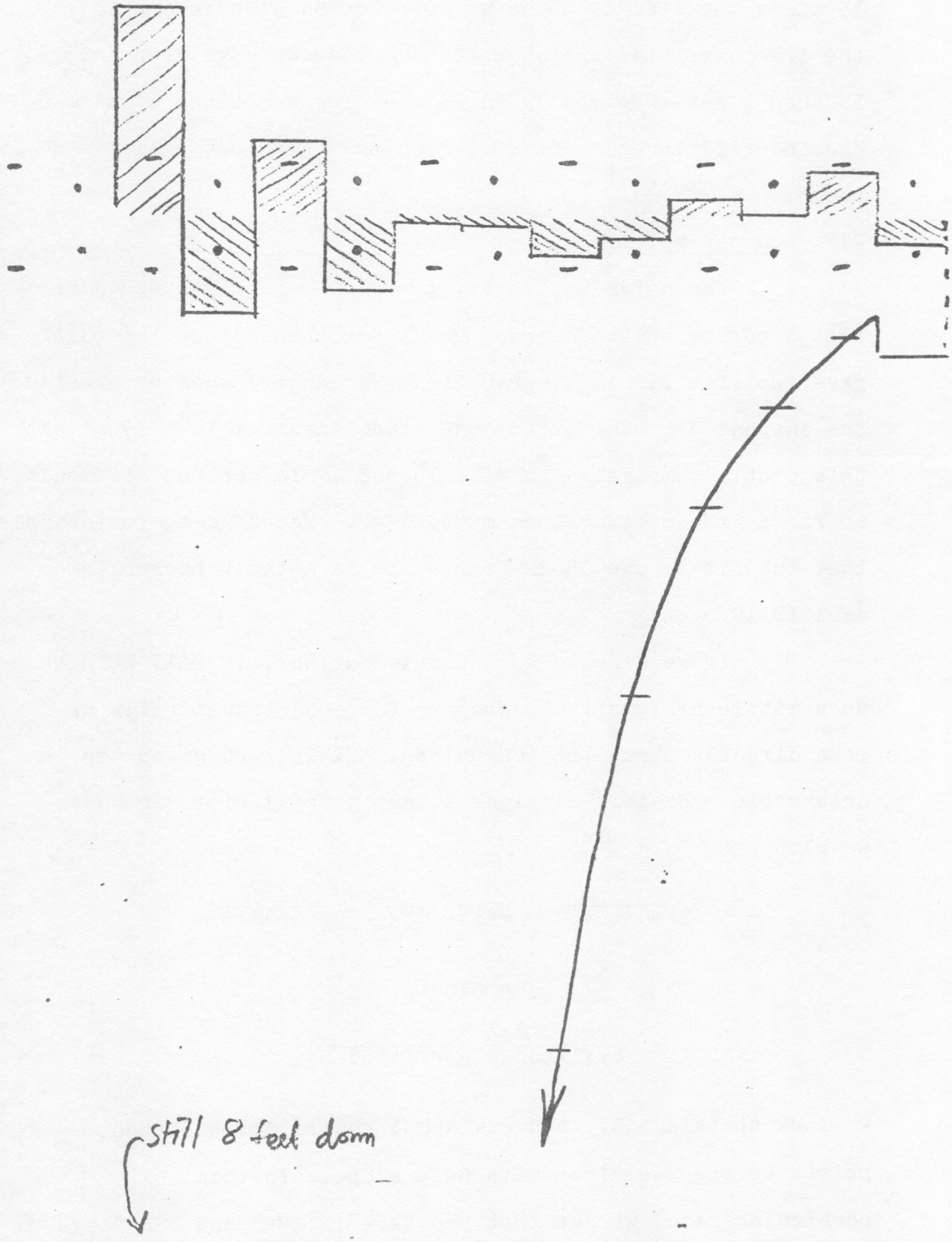


exhibit 20 of chapter 26 -- marriage ages

Comparison-of-fits arithmetic for 235,252 ages of spinsters at marriage

A) ARITHMETIC

Age range	Counts*			Simple roots*				
	Obs'd	Fit A	Fit B	Obs'd	Fit A	Fit B	■	▲
15-	17546	15969	16448	132.4	126.4	128.3	-1.9	10.1
20-	118542	120458	120061	344.3	347.1	346.5	-.6	-5.0
25-	70411	69289	69508	264.3	262.2	263.6	1.4	2.8
30-	20241	21009	20907	142.3	144.4	144.6	.2	-4.4
35-	5873	5896	5770	76.6	76.8	76.0	-.8	.4
40-	1706	1732	1692	41.3	41.6	41.1	-.5	-.1
45-	636	582	544	25.2	24.1	23.3	-1.0	3.2
50-	171	191	191	13.1	13.8	13.8	0	-1.4
55-	64	71	73	8.0	8.4	8.5	.9	-.1
60-	28	28	30	5.2	5.2	5.5	.3	-.3
65-	23	12	13	4.8	3.5	3.6	.1	2.5
70-	11	16	13	3.3	4.0	3.6	-.4	-1.0

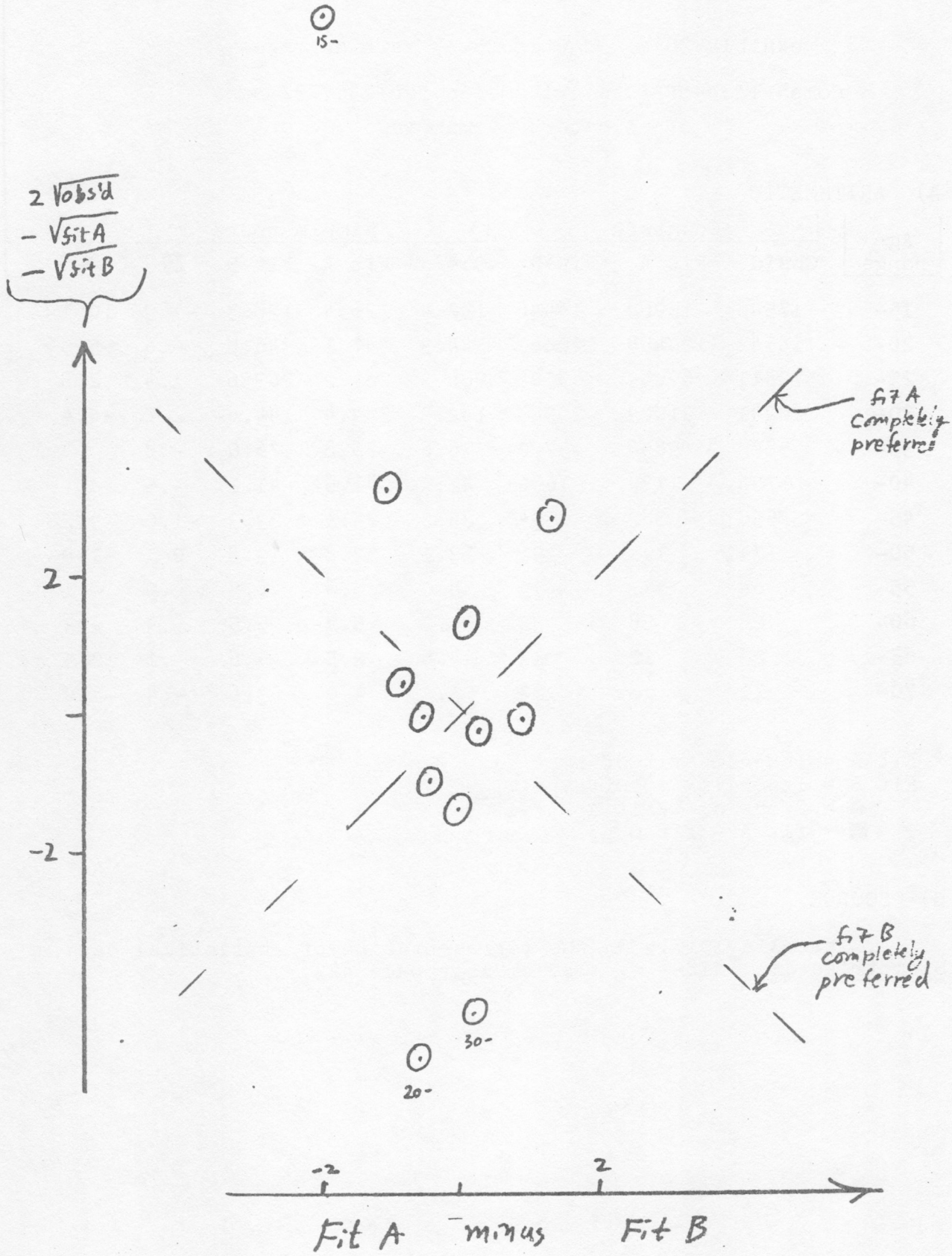
* Fit A = Isserlis fit of type VI.
Fit B = Isserlis fit of type V.
▲ = $2\sqrt{\text{obs'd}} - \sqrt{\text{fit A}} - \sqrt{\text{fit B}}$
■ = fit A - fit B

B) SOURCE

L. Isserlis 1917. On the representation of statistical data. Biometrika 11: 418- . Table VI at page 423.

exhibit 21 of chapter 26 -- marriage ages

Comparison-of-fits plot



does somewhat better for "15-" -- and that the other age classes combine to make a toss-up.

Exhibit 22 gives the arithmetic for another instance. Exhibit 23 compares fit A and fit B. Again, both fits deviate from the observed values in rather similar ways, as is shown by the large ratio of ranges

$$\frac{4.43}{.61} = 7.2$$

(This ratio would be 1 if either fit exactly equaled observed.)

In this case, however -- as shown either graphically by exhibit 23 or arithmetically by panel B of exhibit 22 -- there is a definite tilt to the points of the plot. The only thing to bother us is that this tilt is about three times as steep as would correspond to choosing fit B.

If we must take one or the other, we clearly take fit B.

If we are allowed linear combinations, we are led to look at

$$2 \cdot \text{fit B} - 1 \cdot \text{fit A}$$

since this is as far beyond fit B as fit B is beyond fit A.

For further background, exhibit 24 shows the comparison of McGuire, Brindley, and Bancroft's fit C with the data of exhibit 22. Notice that since the number of larvae can only be an integer, we ought to show the fit as a staircase and not as a smooth curve. The fit is obviously quite good.

Comparison-of-fits arithmetic for 3205 counts of corn borer larvae in field corn

A) ARITHMETIC

Borers per plant	Counts				Simple roots				* ▲	* ■
	Obs'd	Fit A	Fit B	Fit C	Obs'd	Fit A	Fit B	Fit C		
0	588	553.18	568.14	587.85	24.25	23.52	23.83	24.25	-.31	1.15
1	807	845.72	829.36	799.38	28.41	29.08	28.80	28.27	.28	-1.06
2	741	750.81	742.53	741.12	27.22	27.40	27.25	27.22	.15	-.21
3	479	506.58	509.95	515.08	21.89	22.51	22.58	22.70	-.07	-1.31
4	328	287.83	293.57	299.67	18.11	16.97	17.13	17.31	-.16	2.12
5	159	145.14	148.37	150.76	12.61	12.04	12.18	12.28	-.14	1.00
6	67	67.01	67.71	67.74	8.19	8.19	8.23	8.23	-.04	-.04
7	22	28.89	28.43	27.64	4.69	5.37	5.33	5.26	.04	-1.32
8	5	11.80	11.13	10.39	2.24	3.45	3.34	3.22	.11	-2.31
9	7	4.61	4.11	3.63	2.65	2.16	2.03	1.91	.13	1.11
10	2	2.76	1.92	1.74	1.41	1.66	1.39	1.32	.27	-.23
					(Range)				(.61)	(4.43)

* ■ = $2\sqrt{\text{obs'd}} - \sqrt{\text{fit A}} - \sqrt{\text{fit B}}$
▲ = $\sqrt{\text{fit A}} - \sqrt{\text{fit B}}$

B) COMPARISON of FITS A and B -- reordered values

Reordered pairs ▲	Reordered pairs ■	End medians ▲	End medians ■
-.34	1.12		
-.17	2.13	-.17	1.12
-.14	1.00		
-.07	-1.31		
-.04	-.04		
.04	-1.32		
.10	-2.30		
.11	1.13		
.15	-.21		
.27	-.23	-.27	-.23
.28	-1.06		

slope

$$\frac{-1.35}{.44} = -3.1$$

(cont'd)

exhibit 22 (Cont'd)

C) PROBLEMS

1) Compare fits A and C, both arithmetically and graphically. Describe your results.

2) Compare fits B and C, both arithmetically and graphically. Describe your results.

3) Calculate values of "2·fit B - 1·fit A" and compare with fit C. Describe your results.

4) Repeat the calculation of one simple root column, using fits to the nearest 0.1.

D) SOURCE

J. U. McGuire, T. A. Brindley, and T. A. Bancroft 1957. The distribution of European corn borer larvae, Pyrausta Nubilalis (Hbn.), in field corn. Biometrics 13:65-78. Distribution 2 on page 75.

exhibit 23 of chapter 26 -- corn borers

Comparison-of-fit plot for corn-borer larvae
(see exhibit 22)

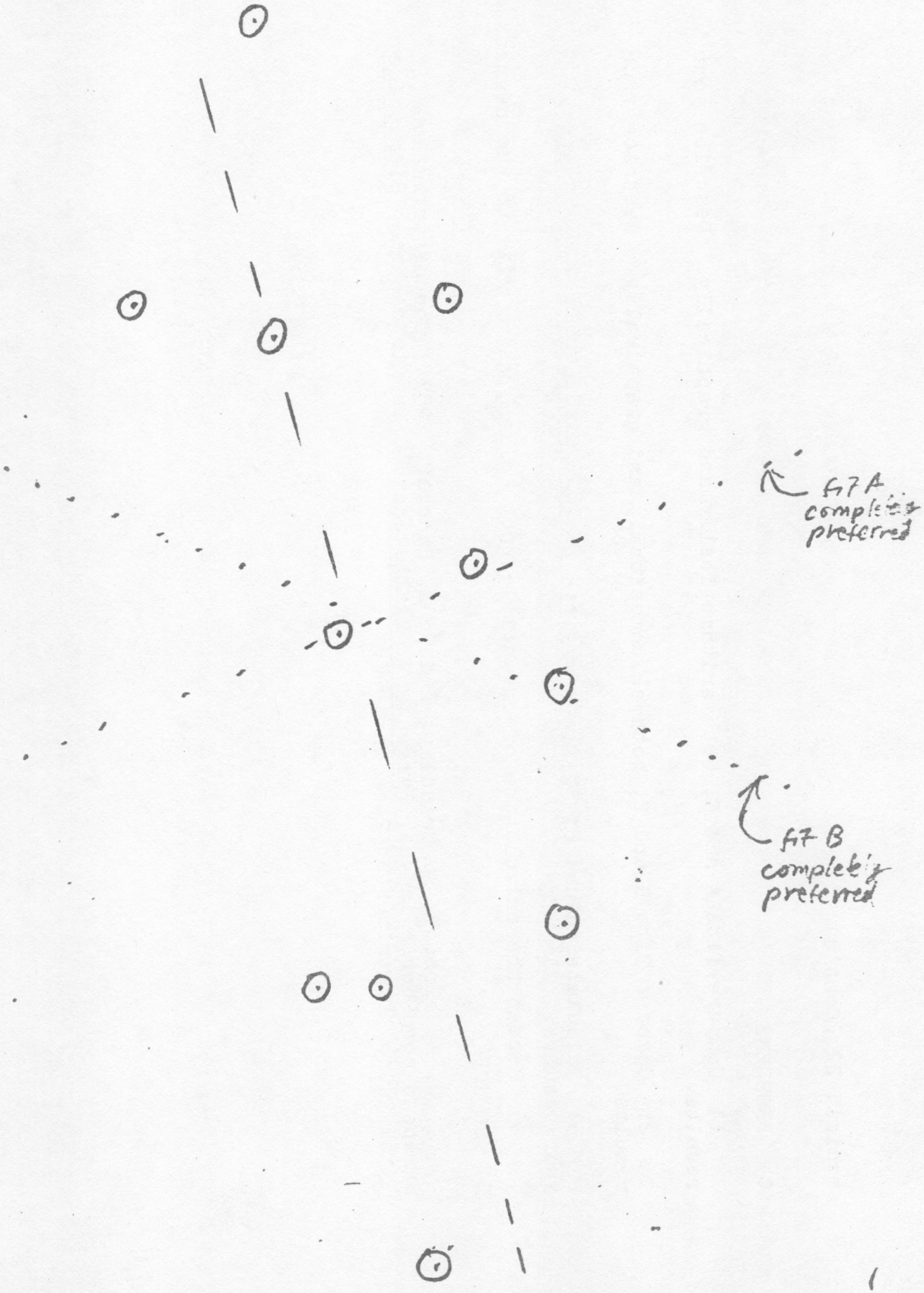
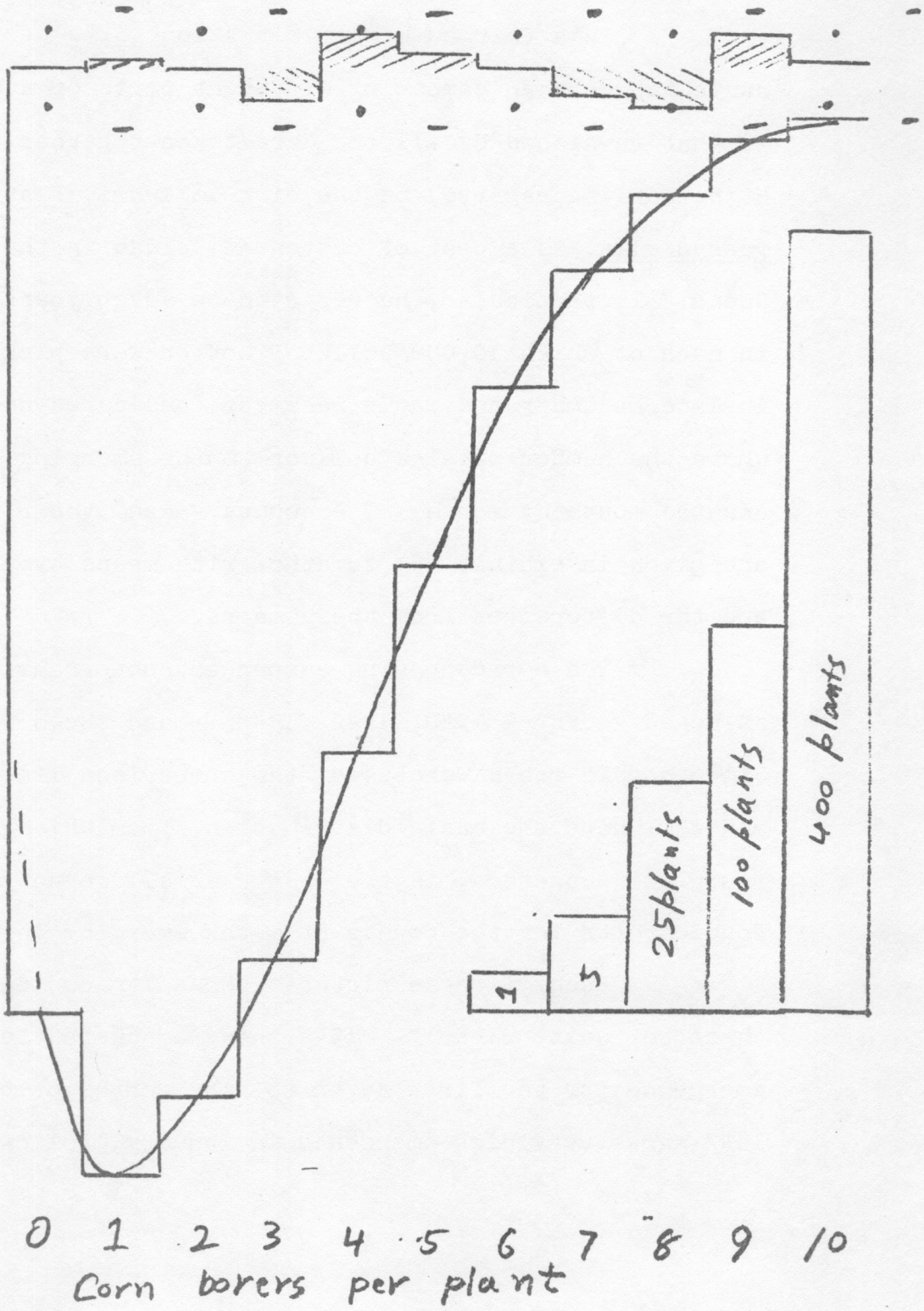


exhibit 24 of chapter 26 -- corn borers

Corn borers in field corn; 3205 counts

(Fit c by McGuire, Brindley and Bancroft, see exhibit 22)



We will have more to say about suspended rootograms when what is counted is a count in a latter chapter. (When we come to section 27B.)

26G. Comparison of one with all

In a fair number of situations, the best comparison curve to use when comparing different parts of a body of data is that furnished by all the data taken together. In 1951, H. S. Darling reported on the pink bollworm, *Platyedra gossypiella*, as a pest of cotton at Zeidab in the Northern Sudan. In particular, he reported on seven years of experiments in each of which 10,000 bolls of cotton were picked at Zeidab in late November and early December, and caged under conditions where the number of pink bollworm moths emerging could be counted month by month. The counts -- and their double roots -- are given in exhibit 25, together with means over seven years and the differences from these means.

The corresponding suspended rootograms for three selected years -- 1940, 1942, 1948 -- are shown in exhibit 26. Since double roots were used, the comparison lines are at ±2 and ±3. Here the basic distribution, from which the square-roots are suspended, is the mean over all seven years of the double roots for the counts of moths emerging in a given month.

Each of these pictures shows its own distinct character quite clearly. 1948 shows a substantial surplus emergence for the first seven to nine months of the year. 1942 shows very high emergence in January, followed by smaller,

exhibit 25 of chapter 26 -- bollworm moths

Numbers of pink bollworm moths emerging each month
from 10,000 bolls of cotton

A) COUNTS

Month	1940	1941	1942	1943	1944	(*)	1948	1949
Jan	96	68	285	100	40		203	12
Feb	57	39	44	74	96		89	8
Mar	71	47	25	20	46		116	18
Apr	38	8	47	60	20		79	21
May	0	0	28	52	16		42	29
June	0	0	10	13	1		14	0
July	0	0	1	18	2		7	0
Aug	0	0	4	19	1		18	4
Sept	0	0	0	2	0		3	1
Oct	0	0	0	1	0		0	0

B) SQUARE ROOTS -- double roots from exhibit 2 of chapter 24

								(**)
Jan	19.6	16.6	33.8	20.0	12.7		28.5	7.1	(19.7)
Feb	15.2	12.6	13.3	17.3	19.6		18.9	5.8	(16.1)
Mar	16.9	13.8	10.1	9.0	13.6		21.6	8.6	(13.4)
Apr	12.4	5.8	13.8	15.6	9.0		17.8	9.2	(13.9)
May	1.0	1.0	10.6	14.5	8.1		13.0	10.8	(8.4)
June	1.0	1.0	6.5	7.3	2.4		7.6	1.0	(3.8)
July	1.0	1.0	2.4	8.6	3.1		5.5	1.0	(3.2)
Aug	1.0	1.0	4.2	8.8	2.4		8.6	4.2	(3.1)
Sept	1.0	1.0	1.0	3.1	1.0		3.7	2.4	(2.0)
Oct	1.0	1.0	1.0	2.4	1.0		1.0	1.0	(1.2)

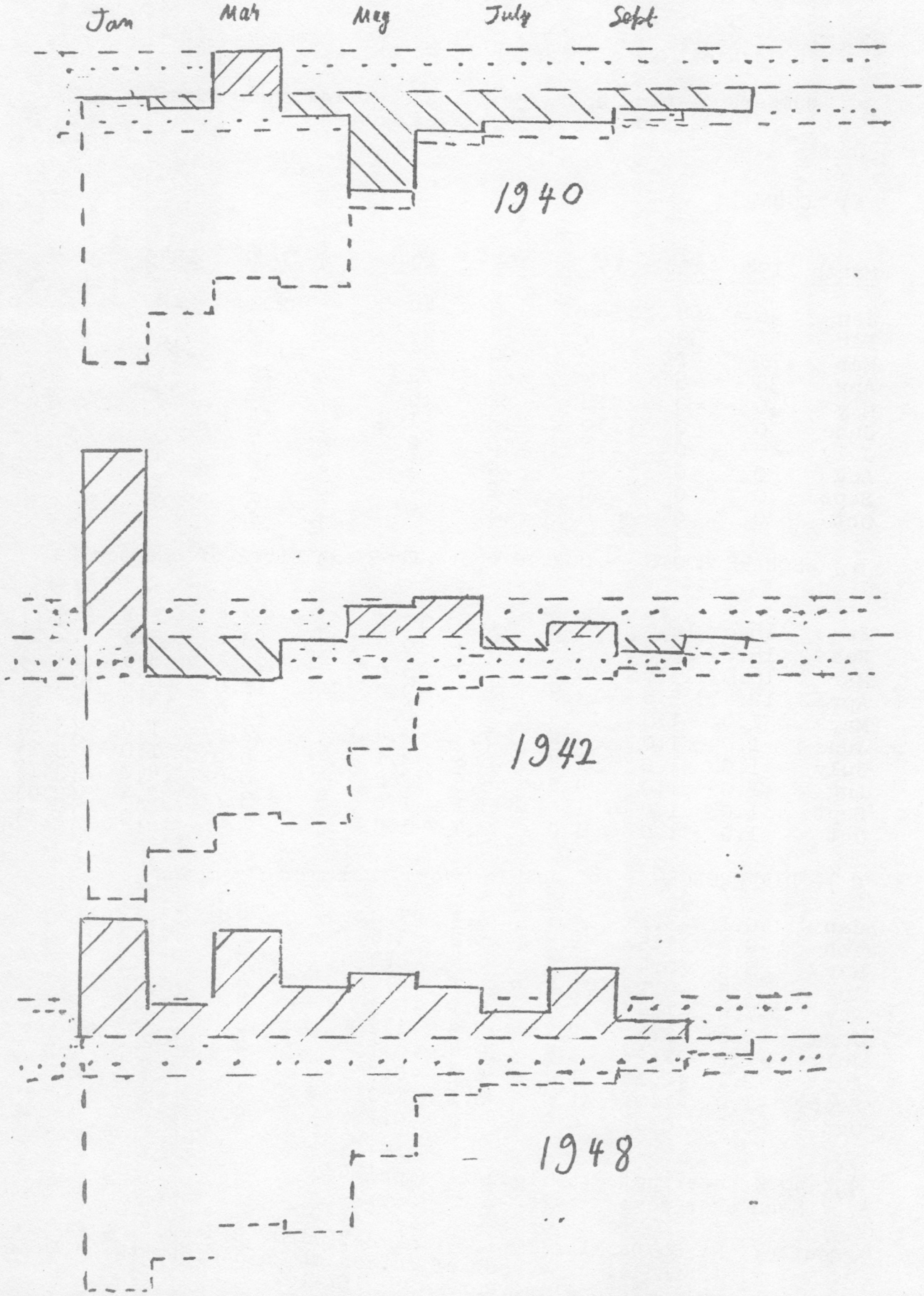
C) DIFFERENCES -- of double roots from monthly means

Jan	-0.1	-3.1	14.1	0.3	-7.0	8.8	-12.8
Feb	-0.9	-3.5	-2.8	1.2	3.6	2.8	-10.3
Mar	3.5	0.4	-3.3	-4.4	0.2	8.2	-4.8
Apr	-1.5	-8.1	-0.1	-1.7	-4.9	3.9	-4.7
May	-7.4	-7.4	2.2	6.1	-0.3	4.6	2.4
June	-2.8	-2.6	2.7	3.5	-1.4	3.8	-2.8
July	-2.2	-2.2	-0.8	5.4	-0.1	2.3	-2.8
Aug	-2.1	-2.1	1.1	5.7	-0.7	5.5	1.1
Sept	-1.0	-1.0	-1.0	1.1	-1.0	1.7	0.4
Oct	-0.2	-0.2	-0.2	1.2	-0.2	-0.2	-0.2

(*) Work interrupted by locust campaigns.
(**) Mean over seven years.

Source: H. S. Darling 1951. Pink Bollworm, *Platyedra gossypiella* (Saund.), as a pest of cotton at Zeidab, Northern Sudan. Bull. Entomolog. Res. 42: 157-167.

Distribution of pink bollworm moths by month of emergence
(---- is mean of double roots for seven years, see exhibit 25)



more or less oscillatory, deficits and surpluses -- which may be quite accidental. 1940 shows a tendency to a strong deficit from May onward. It is very hard to see as much from any table of numbers as we can easily see from these pictures.

*****

26H. How far have we come?

This chapter has dealt with the display of patterns of counts -- distributions of values -- in explicit comparison with a reference pattern, often a smooth comparison curve. We began by noticing that the displays of the last chapter -- because people as viewers are as they are -- involve an implicit comparison. Even though no trace of its presence be on the paper, a rectangular distribution is implicitly present in such displays.

For most patterns of counts per unit interval, a humpy, long-drawn-out reference is better. It needs to be explicit, since there are so many different possibilities. Its use is helpful even when we can see that a different humpy standard would be more clearly matched by the data.

Comparisons, of points or of ends of bars, with straight lines are better than with curves. Accordingly, we have learned to hang or suspend the bars of our histograms, rootograms or rootobars.

Some of the advantage of rootobars remains when they are hung or suspended, but the great advantage of the area principle seems to be confined to those situations where there is no other explicit standard of comparison. Accordingly,

suspended or hanging rootograms

seem to be the display of choice, with suspended or hanging rootobars coming next, still ahead of suspended or hanging histograms.

When, as often, we need to fit the comparison curve to data, we lose little if we are a little crude in our fitting. The display will usually still show what we need most to see. Putting the hinges in about the right place is usually good enough. We are now prepared to do this readily for rectangular, Gaussian, and Cauchy comparison curves and for either histograms or rootograms.

Experience with good fits encourages us to put comparison lines on all our suspended rootograms. (At ±1 and ±1.5 for simple roots; at ±2 and ±3 for double roots.)

Other people's fits can be looked at easily with suspended rootograms, too.

Rather often the best basis for comparison is not some theoretico-mathematical curve, but rather the total behavior of several patterns of counts -- patterns which are, in some useful sense, parallel to one another. Here again we can use the principles of

- suspension or hanging.
- square-roots of counts.

Comparison of patterns of counts boils down to a few essentials:

- reasonable wisdom in the choice of the standard of comparison.
- hanging or suspension so that detailed comparisons are relative to a straight line.
- using square roots, perhaps even logs, of counts.
- keeping the displays informative yet with minimum confusion.

*****

26K. Treating small fitted counts better

Just using roots -- whether simple or double -- is not enough when both observed and fitted counts are quite small. We have quite different thoughts about observing a count of 2 when the fitted count is 0.01, 0.1, or 0.5, yet the root residuals are quite similar, namely

$$\sqrt{2} - \sqrt{0.01} = 1.3$$
$$\sqrt{2} - \sqrt{0.1} = 1.1$$
$$\sqrt{2} - \sqrt{0.5} = 0.7$$

for simple root and

$$\sqrt{10} - \sqrt{1.04} = 2.1$$
$$\sqrt{10} - \sqrt{1.4} = 2.0$$
$$\sqrt{10} - \sqrt{3} = 1.4$$

for double roots.

There are times when we could make good use of something better. Exhibit 27 provides it in three forms, depending on whether:

- we are using simple roots.
- we are using double roots, but have not yet found a double-root residual.
- we are using double roots, and have already found a double-root residual.

As an example, let us turn back to exhibits 17 and 16, where one extreme class seemed rather deviant, the fitted

exhibit 27 of chapter 26 -- tables

More appropriate residuals for use with small fitted counts

A) When SIMPLE ROOTS are used -- replacements

fitted count	Observed Count												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0.001	.0	1.6	2.5	3.2	3.8								
0.002	.0	1.5	2.4	3.0	3.6								
0.005	.0	1.4	2.2	2.8	3.4	3.8							
0.01	.0	1.3	2.0	2.6	3.0	3.6	4.0	4.6					
0.02	.0	1.2	1.9	2.4	2.9	3.4	3.8	4.1	4.4				
0.05	[illegible]	1.0	1.6	2.2	2.6	3.0	3.4	3.6	3.8				
0.1	.0	.8	1.4	1.9	2.3	2.7	3.0	3.4	3.6	4.0	4.2		
0.2	-.1	.6	1.2	1.6	2.4	2.4	2.7	3.0	3.2	3.6	3.8		
0.4	-.2	.4	.9	1.3	1.6	2.0	2.4	2.6	2.8	3.1	3.4	3.6	3.8
0.6	-.3	.3	.8	1.1	1.4	1.8	2.0	2.3	2.6	2.8	3.0	3.3	3.6
0.8	-.4	.2	.6	1.0	1.3	1.6	1.8	2.1	2.4	2.6	2.8	3.0	3.2
1.0	-.5	.0	.4	.8	1.2	1.4	1.7	2.0	2.2	2.4	2.6	2.8	3.0
1.5	-.6	-.1	.2	.6	.8	1.2	1.4	1.6	1.8	2.1	2.3	2.5	2.7
2.0	-.7	-.3	.0	.4	.6	0.9	1.2	1.4	1.6	1.8	2.0	2.2	2.4

Examples: Fitted count = 0.075, observed count = 2, residual to be 1.5.
Fitted count = 1.75, observed count = 10, residual to be 2.2.

(Cont'd)

exhibit 27 (Cont'd)

B) When DOUBLE ROOTS are used -- replacements

fitted count	Observed Count												
	0	1	2	3	4	5	6	7	8	9	10	11	12
0.001	.0	3.3	5.0	6.4	7.6								
0.002	.0	3.1	4.8	6.1	7.2								
0.005	.0	2.8	4.4	5.6	6.7	7.6							
0.01	.0	2.6	4.1	5.1	6.1	7.2	8.1	9.1					
0.02	.0	2.3	3.7	4.8	5.8	6.7	7.5	8.2	8.8				
0.05	-.1	2.0	3.2	4.3	5.2	6.0	6.7	7.2	7.7				
0.1	-.1	1.6	2.8	3.8	4.6	5.4	6.1	6.7	7.3	7.9	8.5		
0.2	-.2	1.3	2.4	3.2	4.0	4.7	5.4	6.0	6.5	7.1	7.6		
0.4	-.4	.9	1.8	2.6	3.3	4.0	4.7	5.2	5.7	6.2	6.7	7.2	7.6
0.6	-.6	.6	1.5	2.2	2.9	3.5	4.1	4.6	5.1	5.6	6.1	6.6	7.1
0.8	-.8	.3	1.2	1.9	2.6	3.2	3.7	4.2	4.7	5.2	5.7	6.1	6.5
1.0	-.9	.1	.9	1.6	2.3	2.9	3.4	3.9	4.4	4.8	5.3	5.7	6.1
1.5	-1.2	-.2	.5	1.1	1.7	2.3	2.8	3.3	3.7	4.2	4.6	3.0	5.4
2.0	-1.5	-.6	.1	.7	1.3	1.8	2.3	2.8	3.2	3.6	4.0	4.4	4.8

Examples: Fitted count = 0.3, observed count = 3, residual to be 3.0.
Fitted count = 0.013, observed count = 1, residual to be 2.8.

(Cont'd)

exhibit 27 (Cont'd)

C) When DOUBLE ROOTS are used -- adjustments

fitted count	Observed Count 0	1	2	3	4	5	6	7	8	9	10	11	12
.001	-.4	1.8	2.9	3.7	4.3								
.002	-.4	1.6	2.6	3.3	4.0								
.005	-.4	1.4	2.2	2.9	3.4	3.9							
0.01	-.4	1.2	1.9	2.4	2.9	3.5							
0.02	-.4	.9	1.6	2.1	2.6	3.0	3.4						
0.05	-.3	.6	1.2	1.6	2.0	2.4	2.7	2.7					
0.1	-.3	.4	.8	1.2	1.6	1.9	2.2	2.3	2.7	2.9	3.2		
0.2	-.2	.2	.5	.8	1.1	1.4	1.5	1.8	2.0	2.3	2.5		
0.4	-.2	.0	.3	.5	.7	.9	1.0	1.3	1.5	1.6	1.8	2.0	2.1
0.6	-.2	.0	.1	.4	.6	.8	.8	1.0	1.1	1.3	1.5	1.7	1.9
0.8	-.2	-.1	.1	.2	.4	.5	.6	.8	1.0	1.1	1.3	1.4	1.5
1.0	-.1	-.1	.0	.2	.3	.4	.5	.7	.8	.9	1.1	1.2	1.3
1.5	.0	-.1	.0	.0	.1	.2	.3	.4	.6	.6	.8	.9	1.0
2.0	.1	-.1	-.1	.0	.0	.1	.2	.3	.4	.5	.6	.6	.7

Examples: Fitted count = 0.1, observed count = 4, raw double-root residual = $\sqrt{18} - \sqrt{1.4} = 3.06$ (already calculated), table gives 1.6, residual to be 3.1+1.6 = 4.7.

Fitted count = 0.8, observed count = 8, raw double-root residual = $\sqrt{34} - \sqrt{4.2} = 3.79$ (already calculated), table gives 1.0, residual to be 3.8+1.0 = 4.8.

count being 0.5 and the observed count being 4.5. Reference to panel A shows that a simple-root residual of 1.7 is more appropriate than the residual of 1.4 found by subtracting $\sqrt{0.5}$ from $\sqrt{4.5}$.

*****

26L. How closely does the comparison curve need to be fitted?

Close to the start of section 26D we were a little arbitrary. The hinges of our 120 residuals were so placed that the intervals corresponding to the units digit of the residuals were about 0.59 long, and the boundary between the center two intervals came at about +0.06, all in units of the half midspread. We replaced 0.59 by 0.60 and 0.06 by 0.00 with very little discussion.

Did these shifts matter? What size shifts can we safely make -- for pure convenience -- in other situations? How are we to notice that the fit we are using needs adjustment?

When we are looking for individual abnormalities, the answer to this question tends to be: No, it doesn't matter. Why should this be so?

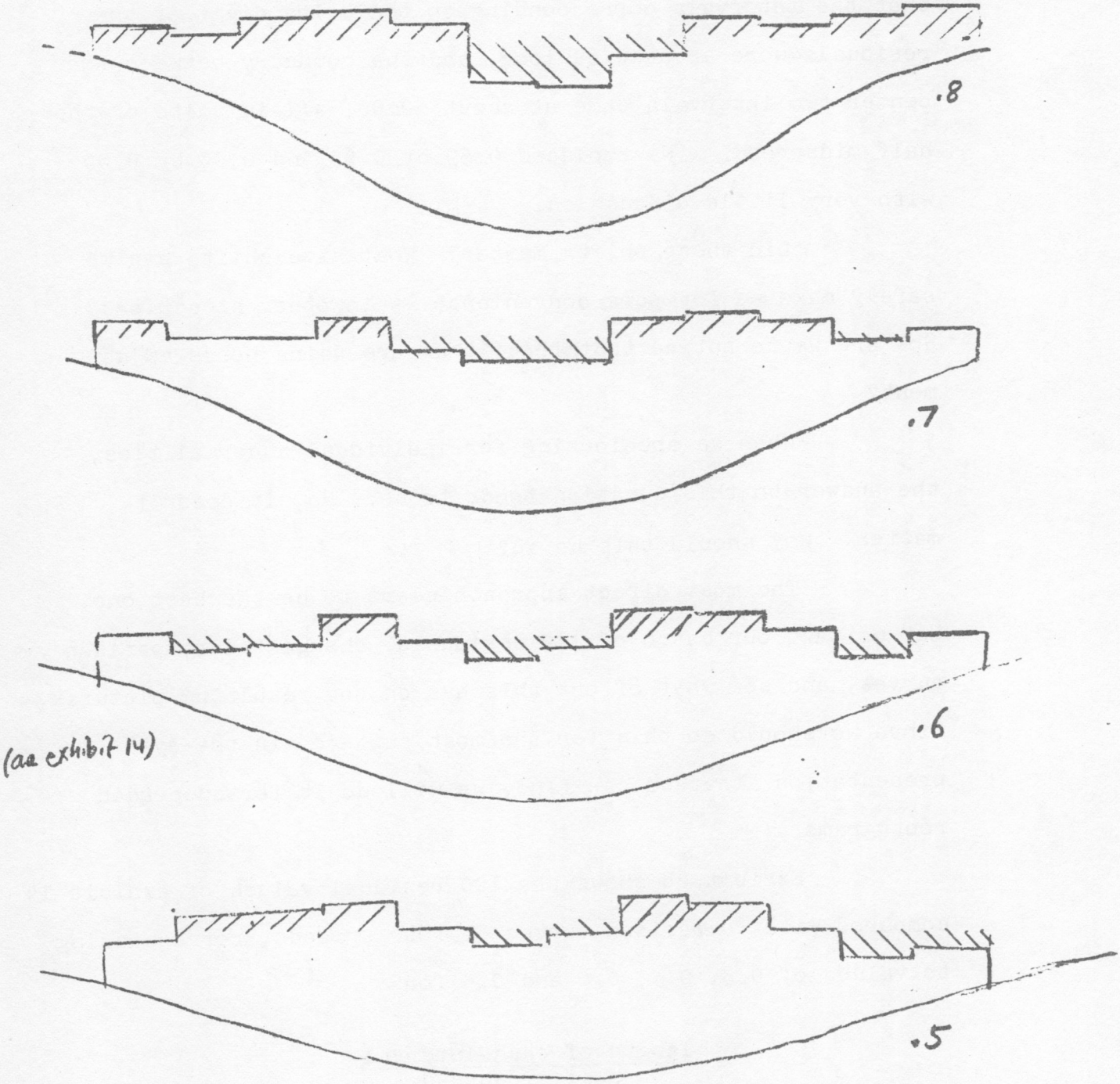
The most direct approach seems to be the best one. Let us take our batch of 120 residuals, change the comparison curves, and see what effect this has on the resulting pictures. Since we should do this for the most "hit-'em-in-the-eye" presentation that we can find, we will do it for suspended rootograms.

Exhibit 28 shows the 120 residual values of exhibit 14 compared with comparison curves of four spreads, corresponding to values of 0.8, 0.7, 0.6 and 0.5 for

$$\frac{\text{length of each interval}}{\text{half midspread}}$$

exhibit 28 of chapter 26 -- Nebraska residuals

Suspended rootograms comparing the same observations with Gaussian comparison curves with different spreads. Labels give length of intervals in half-midspreads



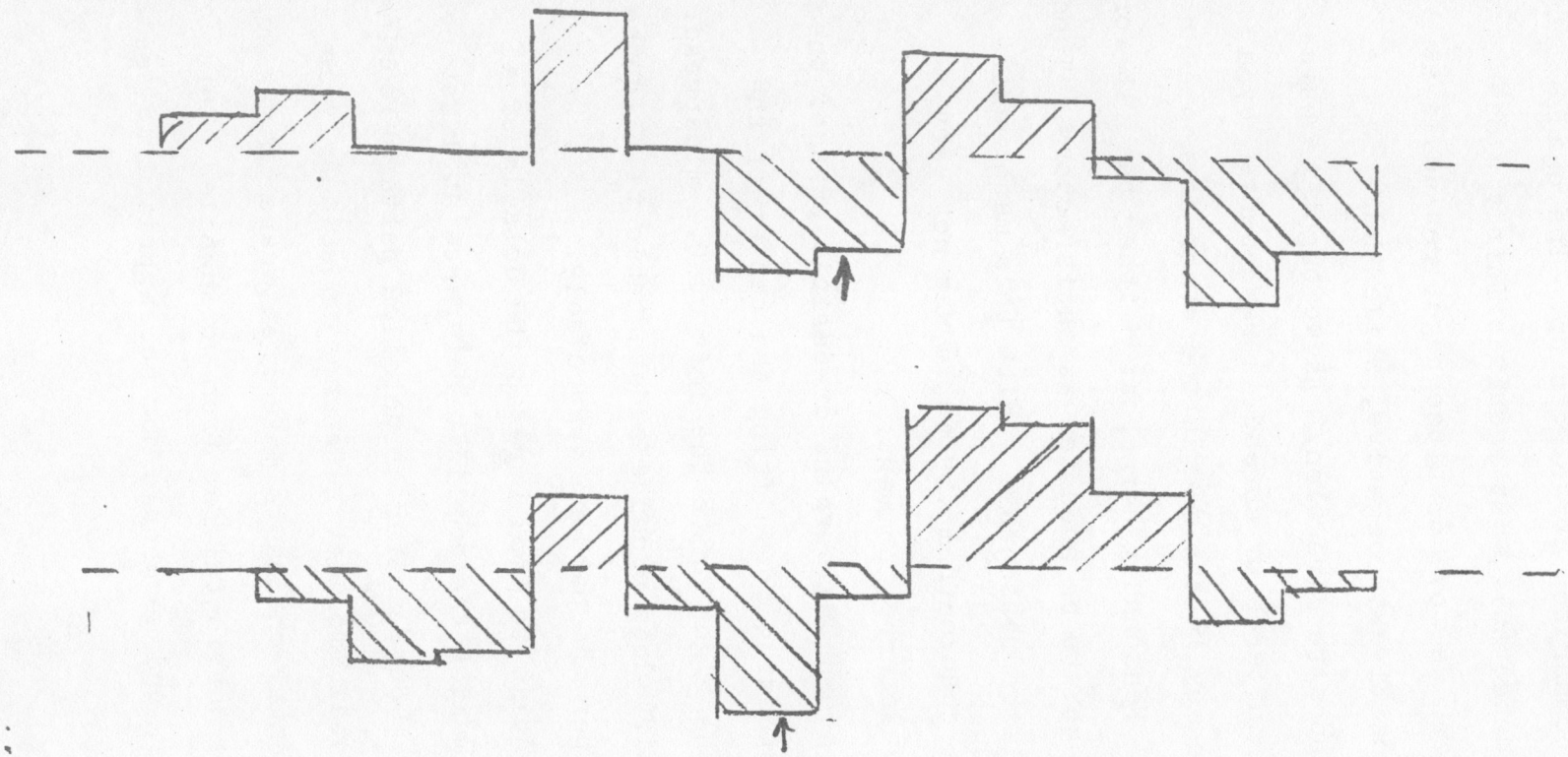
As this ratio goes down, the number of intervals need to cover a given part of the comparison curve goes up. The change from each to the next is detectable, but requires fairly close attention. Overall, as we move down the page, residuals at the ends move from positive to negative, while those in the middle move in the opposite direction. As we ought to have expected, a poor fit in terms of spread is represented by a general bowing -- one way in the middle the other at the ends.

We can feel reasonably sure that neither the fit for .5 nor that for .8 ought to be accepted as satisfactory. Those for .7 and .6 may well be reasonable. With this much uncertainty remaining, the change from .59 to .6 was not very meaningful and hardly deserves comment.

Exhibit 29 shows the surplus-or-deficit part of the two rootograms corresponding to shifting the center of the comparison curve to the right or to the left by 0.2 midspread. We see that such a shift does produce a detectable tilt in the plot, but does not alter the pattern appreciably.

If our intent is to look hard at the details of a suspended rootogram, making the implicit reference at least nearly horizontal is helpful. But, so far as detailed pattern-looking goes, a nearly horizontal implicit reference is so nearly as good as a quite horizontal one as to make great care in making the implicit reference horizontal far from worthwhile. This is a general principle, and we should not be surprised to see it working here too.

exhibit 29 of chapter 26 -- Nebraska residuals



Effect on deficit and surplus of shifting the comparison curve sidewise by ±0.2 in units of the half-midspread.
(Arrows show location of center of comparison curve.)

Shifts of ±0.2 were detectable as shifts, but did nothing to distort the picture. Taking +0.06 as 0.00 was thus a wholly negligible change.

When we are trying to evaluate the overall quality of fit, we might expect to need much higher standards of fitting. Yet a careful look at either exhibit 27 or exhibit 28 shows us that the apparent quality of fit doesn't change very fast as we move the conversion around. We need more care than for looking for anomalies -- but probably not as much care as we took in section 26D. (If we had 235,252 or so observations, we might need to be a bit more careful.)

If we want to be really serious about our comparison lines, we may need to be a little more careful about our choice of conversion.

*****

26M. How to make extra-refined fits (optional)

There are a number of points about the way that we have chosen to find comparison curves to which a purist might object. These might include:

- doubt of the adequacy of taking the height of the curve at the middle of the interval as the ratio of area to base.

- unhappiness about exactly how one chooses which intervals are to be considered, and about what ought to be done with the tailpieces that remain.

- wonder whether more decimals in the tables of exhibits 10 and 13 might be worthwhile.

- questioning of our use of hinges alone to fix the spread and location of the comparison curve.

This section will provide us techniques for dealing with the first two of these four. The third will be discussed briefly in a moment. The fourth -- and last -- ought to be met head on by an inquiry into just what we are trying to do. We will turn to this shortly.

* more decimals? *

From our point of view, rootograms are more meaningful than histograms, so let us turn to the precision of exhibit 13. We are using "h" where it seems appropriate, so that we could hold our error down to about ±.003, if we are not careless in assembling the table. If the result in $\sqrt{\ }$'s is not more than, say, ±0.3 in error, we are not in serious trouble.

Let us look at a few illustrative situations where an error of ±.003 in the tabled value will give an error of ±0.3 in the fitted value. Before looking generally, it will pay to consider one case in detail. Suppose the tabular entry is .50. A change of ±.003 will change the fitted root value by ±0.3 if that fitted root is 50, corresponding to a count of 2500. This is for one cell, not for the total. To find the total we need to multiply the cell total by

$$\frac{\ldots + (\ )^2 + (\ )^2 + (.50)^2 + (\ )^2 + (\ )^2 + \ldots}{(.50)^2}$$

where the sum in the numerator is over the squares of the tabled roots for all cells. Reasonable values for this sum are:

3, for very few cells.

6, for moderately few cells.

12, for more cells.

The corresponding ratios in our example are 3/.25 = 12, 6/.25 = 24, and 12/.25 = 48, leading to total counts of 30,000, 60,000 and 120,000. We are not very likely to have this many counts. Thus we are likely to do quite well near .50.

A few other instances may help, such as these:

Gauss roots		Equivalent count			
Tabled	**Fitted**	**Cell**	**For 3**	**For 6**	**For 12**
1.00	100	10,000	30,000	60,000	120,000
.50	50	2,500	30,000	60,000	120,000
.20	20	400	30,000	60,000	120,000
.10	10	100	30,000	60,000	120,000

In every case it would require equally large total counts before we could be off by ±0.3 in our roots. When, as, and if we get such large collections we can consider thinking of some other approach.

* better ways of fitting? *

Above all, with our general philosophy, we want to have our choice of comparison curve resistant. This means that we dare not go to the classical measures -- like the mean -- as a basis for fitting. There are no doubt better things to do than to use only the hinges. It is not yet clear what they are. Until they appear, we will do well to use the hinges but be receptive to change.

* improving on mid-ordinates *

We have planned to take the height of curve at the middle of the interval as proportional to the corresponding area. This is usually good enough, but sometimes we have to deal with very long intervals. In such a situation, we can use a formula derived from what is often called Simpson's Rule, taking the area proportional to:

(height at one end of interval)

+ 4·(height in middle)

+ (height at other end)

(We could divide by 6, but, in our situation, the scaling factor will take care of this at no extra cost.)

Exhibit 30 illustrates this for the intervals of exhibit 11.

If we are interested in a rootogram, we can calculate for a histogram and then take square roots, as will shortly be illustrated.

* taking care of the ends *

If we want to do a more careful job of finding the

$$\frac{\text{fitted}}{\text{tabled}}$$

factor, we need:

- first, to allow for the part of the area under the comparison curve not included in any of the chosen intervals by converting this part into counts. (For the case of exhibit 11, for example, this is found, using exhibit 10 to be .004 of 120 or 0.5 cases.)
- second, to divide the remainder, here 119.5 out of 120, by the sum of the tabled heights in order to find the scaling factor.

The details are set out in exhibit 31. Notice that the sizes of the intervals with few counts have been carried to more decimals in order that their square roots be well defined.

The last three columns of square roots give the numbers that could have been plotted in exhibit 3 through 8. (Note minor changes in areas calculated for the curve.)

exhibit 30 of chapter 26 -- Nebraska residuals

Simpson calculation for the intervals and comparison curve of exhibit 11

A) ARITHMETIC

Converted		Gaussian counts		
Centers	**Cut**	**Tabled**	**Simpson**	**Fitted**
	-4.2	.02		
-3.9		.03	.19	.62
	-3.6	.05		
-3.3		.09	.54	1.75
	-3.0	.13		
-2.7		.19	1.16	3.74
	-2.4	.27		
-2.1		.37	2.23	7.22
	-1.8	.48		
-1.5		.60	3.60	11.68
	-1.2	.72		
-0.9		.83	4.96	16.08
	-0.6	.92		
-0.3		.98	5.84	18.90
	0.0	1.00		
0.3		.98	5.84	18.90
	0.6	.92		
0.9		.83	4.96	16.08
	1.2	.72		
1.5		.60	3.60	11.68
	1.8	.48		
2.1		.37	2.23	7.22
	2.4	.27		
2.7		.19	1.16	3.74
	3.0	.13		
3.3		.09	.54	1.75
	3.6	.05		
3.9		.03	.19	.62
	4.2	.02		
	(Sum)		(37.04)	

$$\frac{\text{Fitted}}{\text{Simpson}} \text{ taken as } = \frac{120}{37.04} = 3.24$$

B) PROBLEM

Repeat for a Cauchy comparison curve.

exhibit 31 of chapter 26

Calculation of a Gaussian comparison curve
and its relation to the data of exhibit 5 of chapter 17
(All intervals of equal length)

Center*	Relative height		Square roots		
	Tabled	Fitted	Fitted	Obs'd	Diff.
-3.9	.032	.62	.79	1.0	+.2
-3.3	.087	1.68	1.30	1.4	+.1
-2.7	.194	3.74	1.94	1.7	-.25
-2.1	.37	7.2	2.69	2.45	-.25
-1.5	.60	11.6	3.40	4.0	+.6
-0.9	.83	16.0	4.00	3.9	-.1
-0.3	.98	19.0	4.36	3.6	-.75
+0.3	.98	19.0	4.36	3.9	-.45
+0.9	.83	16.0	4.00	4.7	+.7
+1.5	.60	11.6	3.40	4.0	+.6
+2.1	.37	7.2	2.69	2.8	+.1
+2.7	.194	3.74	1.94	1.4	-.55
+3.3	.087	1.68	1.30	1.0	-.2
+3.9	.032	.62	.79	0.0	-.8
	Sum 5426	119.68			
	76				
	6.186				

* Center of interval, in units of half midspread.

Beyond +4.2 .002
Beyond -4.2 .002
.004

$\frac{\text{Fitted}}{\text{Tabled}}$ taken as $= \frac{119.5}{6.186} = 19.35$

(Reduces 120 to 119.5 = 120 - (.004)(120).)

When we have to deal with unequal bases, we have a few more calculations to make. When these bases are simple multiples of one another, we can use the device shown in exhibit 32.

exhibit 32 of chapter 26

Calculation of a Gaussian comparison curve

with unequal intervals simply related

			Gaussian counts				Square roots		
Center	Widths*		Height	Prod**	Scaled***		Scaled	Obs'd****	Diff.
-3.9	0.6	2	.032	.064	.31	.62	.56	0.7	+1.5
-3.3	0.6	2	.087	.174	.84	1.68	.92	1.0	+0.1
-2.7	0.6	2	.194	.388	1.85	3.70	1.36	1.2	-.15
-2.1	0.6	2	.37	.74	3.6	7.1	1.90	1.7	-.2
-1.5	0.6	2	.60	1.20	5.8	11.6	2.40	2.8	+.4
-0.9	0.6	2	.83	1.66	8.0	16.0	2.74	2.75	.0
-0.3	0.6	2	.98	1.96	9.5	18.9	3.08	2.5	-.6
+0.15	0.3	1	.99	.99	9.6	9.6	3.10	2.8	-0.3
+0.45	0.3	1	.95	.95	9.2	9.2	3.03	2.65	-.4
+0.75	0.3	1	.88	.88	8.5	8.5	2.92	3.9	+1.0
+1.05	0.3	1	.78	.78	7.5	7.5	2.74	2.65	-0.1
+1.35	0.3	1	.66	.66	6.4	6.4	2.52	3.5	+1.0
+1.65	0.3	1	.55	.55	5.3	5.3	2.30	2.0	-0.3
+1.95	0.3	1	.42	.42	4.1	4.1	2.02	2.25	+.25
+2.25	0.3	1	.32	.32	3.1	3.1	1.76	1.7	-.05
+2.55	0.3	1	.21	.21	2.0	2.0	1.41	1.4	.0
+2.85	0.3	1	.160	.160	1.53	1.53	1.23	0.0	-1.2
+3.15	0.3	1	.107	.107	1.01	1.01	1.00	1.0	0.0
+3.45	0.3	1	.069	.069	.66	.66	.81	0.0	-0.8
+3.75	0.3	1	.042	.042	.41	.41	.64	-	-0.6
				8334		8121			
				393		377			
				12.324		118.91			

*Actual and relative widths.

**Product of height and relative width.

***Factor x height (gives counts/unit base) and factor x prod (gives counts/interval).

****Square roots of observed count/width.

culation of ling factor: Beyond -4.2 .002; Beyond +3.9 .005

.007x120 = 0.8

$\frac{119.2}{12.324} = 9.67$ = Factor

*****

26P. Problems for chapter 26

1. Do the problem of exhibit 11.

2. Do the problem of exhibit 12.

3. Do the problem of exhibit 13.

4. Do problem (1) of exhibit 14.

5. Do problem (2) of exhibit 14.

6. Do the problem of exhibit 15.

7. Do the problem of exhibit 16.

8. Do problem (1) of exhibit 22.

9. Do problem (2) of exhibit 22.

10. Do problem (3) of exhibit 22.

11. Do the problem of exhibit 30.

12. Repeat exhibit 31 for a Cauchy comparison curve.

13. Repeat exhibit 32 for a Cauchy comparison curve.

14. In 1957, McGuire, Brindley, and Bancroft published a variety of sets of corn borer counts, and alternative fits. (Their distribution 2 and one of its fits furnished exhibit 20. Exhibit 33 gives their distribution 1 and three fits. Find the corresponding square roots and plot (a) at least one suspended rootogram (b) at least one diagnostic plot comparing this fit with one of the others.

15. In 1906-1953, Elderton published a table counting deaths by age groups and a fitted distribution. Exhibit 34 has the data. Make a suspended rootogram.

16. Turn back to exhibits 7, 8, and 9 of chapter 3 and the distribution of heights of volcanoes. Fit a comparison curve to the heights in each of the three expressions, and plot the suspended rootograms. Compare and discuss.

exhibit 33 of chapter 26

A Corn Borer Distribution and Three Fits

A) DATA AND FITS

Borers per plant	Count of plants			
	Obs'd	Fit A	Fit B	Fit C
0	355	324.30	331.79	341.84
1	600	660.37	654.44	644.37
2	781	734.06	734.77	728.03
3	567	610.45	608.67	609.14
4	441	408.82	411.61	415.60
5	245	236.54	239.64	242.72
6	135	122.49	124.09	125.17
7	42	58.12	58.44	58.20
8	17	25.68	25.45	24.76
9	11	10.70	10.41	9.75
10	11	4.47	5.69	5.42

B) SOURCE

J. U. McGuire, T. A. Brindley, and T. A. Bancroft 1957. The distribution of European corn borer larvae, (Pyraustra nubilalis (Hbn.), in field corn. Biometrics 13: 65-78. Distribution 1 on page 74.

exhibit 34 of chapter 26

Deaths By Age

A) DATA and FIT

Ages	Count of deaths	
	Obs'd	Fitted
30-34	1	1
35-39	5	3
40-44	8	6
45-49	12	14
50-54	28	32
55-59	82	68
60-64	128	137
65-69	253	247
70-74	342	381
75-79	525	480
80-84	438	441
85-89	265	261
90-94	53	80
95-99	18	10
100, etc.	11	1

B) SOURCE

W. P. Elderton 1906-1953. Frequency curves and correlation (1st-4th ed.) page 97 of 4th edition. His source: G. King 19--. On the rate of mortality amongst female nominees,... J. Institute of Actuaries 32: 262-268.

17. In another of his polemics on the adequacy of histograms to show adequacy of fit, Karl Pearson cited the data on greatest skull breadth listed in exhibit 35. Make a suspended rootogram, with comparison lines. Comment on result. .

18. Look up the source for the previous problem and make a suspended rootogram with comparison lines for "greatest forehead breadth" as given in the same table.

19. In his fundamental paper of 1900 on chi-square, Karl Pearson quoted data on 4274 runs on color (lengths of consecutive sequences of "red" or "black") in roulette at Monte Carlo in July 1892. Panel A of exhibit 36 gives the actual data and the counts expected if "red" and "black" behaved like independent throws of a fair coin. Make a suspended histogram with comparison lines. Comment.

20. In the same paper, Karl Pearson gave the counts -- and the non-Gaussian comparison -- given in panel B of exhibit 36. Plot the suspended rootogram. Comment.

21. In the same paper, Karl Pearson used the data in panel C of exhibit 36, and the associated Gaussian comparison, to point out the weakness of the evidence that Merriman had called on to support his desire to analyze data as if every sample came from a normal distribution. Plot a suspended rootogram. Comment.

22. In 1916, Kirstine Smith gave a new fit to a collection of errors in measuring right ascension that had originally been fitted by Bessel. Panel A of exhibit 37 has observations

exhibit 35 of chapter 26

Greatest Skull Breadths in mm

Observed and Fitted Distribution

A) DATA

Breadth (in mm)	Counts Obs'd	Counts Fitted
below 127	2	5.5
127	6	3.7
128	1	5.8
129	9	8.7
130	13	12.9
131	20	18.3
132	28	25.5
133	36	34.3
134	36	45.0
135	58	57.2
136	60	70.6
137	88	84.7
138	108	98.7
139	118	111.6
140	119	122.6
141	143	130.8
142	138	135.5
143	127	136.0
144	140	133.5
145	128	126.4
146	115	116.5
147	115	104.3
148	84	90.6
149	78	76.5
150	52	62.7
151	46	49.9
152	29	38.6
153	30	29.0
154	23	21.1
155	19	15.0
156	8	10.3
157	5	6.6
158	3	4.7
159	5	2.8
160	4	1.7
161	0	1.1
162	1	0.6
163	1	0.3
above 163	1	0.4

B) SOURCE

K. Pearson 1903. Professor Aurel von Torok's attack on the Arithmetical Mean. Biometrika 2:339-345. Table II on page 343.

exhibit 36 of chapter 26

Examples Used by Karl Pearson to Illustrate
His Fundamental Paper of 1900

A) RUNS ON COLOR AT MONTE CARLO -- July 1892

Length:	0	1	2	3	4	5	6	7	8	9	10	11
Obs'd:	2462	945	333	220	135	81	43	30	12	7	5	1
Fitted:	2137	1068	534	267	134	67	33	17	8	4	2	1

His source: K. Pearson 1897. The chances of death and other studies in evolution. Vol. I. The scientific aspect of Monte Carlo Roulette, pages 42-62. See page 54.

His source: Le Monaco (a weekly published in Paris.)

B) NUMBERS OF PETALS IN BUTTERCUPS -- non-Gaussian comparison curve

Petals:	5	6	7	8	9	10	11
Obs'd:	133	55	23	7	2	2	0
Fitted:	136.9	48.5	22.6	9.6	3.4	0.5	0.2

C) 1000 SHOTS AT A LINE ON A TARGET -- counted by bands of equal width parallel to the line

Band:	1	2	3	4	5	6	7	8	9	10	11
Obs'd:	1	4	10	89	190	212	204	193	79	16	2
Fitted:	1	6	27	67	162	242	240	157	70	26	2

His source: M. Merriman 1891. A textbook on the theory of least squares, New York, John Wiley & Sons. See page 14 (still there in the 8th edition, 1915).

His source: Report of the Chief of Ordnance for 1878. Appendix S: Plate VI.

D) SOURCE FOR ALL ABOVE

K. Pearson 1900. On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. Phil. Mag. 50:157-175. Panel A is illustration III on page 169. Panel B is illustration IV on page 169. Panel C is illustration VII on pages 172-173.

exhibit 37 of chapter 26

Examples Used by Kirstine Smith in Her Paper of 1916

A) MEASUREMENTS OF RIGHT ASCENSION

Interval in 0.1 second	Observed (Bradley)	Fitted	
		Bessel	Smith
0-1	114	101.6	98.6
1-2	84	84.1	82.6
2-3	53	57.7	57.9
3-4	24	32.7	34.0
4-5	14	15.4	16.7
5-6	6	6.0	6.9
6-7	3	1.9	2.4
7-8	1	0.5	0.7
8-9	1	0.14	0.21

Note: Fitted counts rounded to nearest 0.1, except for very small values. (More accuracy can hardly help.)
Her source: E. Czuber 1---. Theorie der Beobachtungs fehler. Page 192.

B) CEPHALIC INDEX IN BAVARIAN SKULLS

Index	Observed	Fitted	
		A	B
<75	9.5	12.3	11.4
76	12.5	12.7	12.1
77	17	22.1	21.3
78	37	35.3	34.6
79	55	51.9	51.4
80	71.5	70.1	70.1
81	82	87.0	87.6
82	116	99.4	100.5
83	98	104.2	105.6
84	107	100.5	101.8
85	82	89.1	90.0
86	74	72.6	73.0
87	58	54.3	54.3
88	34.5	37.4	37.0
89	19	23.7	23.1
90	10	13.8	13.3
91	8	7.4	7.0
≥92	9	6.3	6.0

Her source: J. Ranke 1883. Beitrage zur physischen Antropologie der Baiern (Munchen).

C) SOURCE

K. Smith 1916. On the "best" values of the constants in frequency distributions. Biometrika 11:262-276. Panel A from Table I on page 265. Panel B from Table 3 on page 266.

and fits. Make a suspended rootogram for at least one fit. Make a diagnostic plot to compare the fits.

23. In the same paper Smith gave two fits to Ranke's data on the distribution of Cephalic index for Bavarian skulls. Panel B of exhibit 37 has both observed and fitted counts. Proceed as before.

24. Do problem (4) of exhibit 22.